Creative Telescoping and Applications

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January 15, 1996

[summary by Bruno Salvy]

Abstract

Creative telescoping is a method to compute definite sums and integrals. Numerous examples are given, together with an introduction to algorithmic techniques based on Gröbner bases of linear operators.

Creative telescoping applies to solutions of systems of linear recurrences and linear differential equations. It yields a linear recurrence or differential equation satisfied by the definite sum or integral of the solutions. It can be used to "compute" generating functions, to extract their coefficients, and to prove identities.

1. Examples

A typical example is the sum $S_n = \sum_{k=0}^{n} \binom{n}{k}$. One starts with a system of equations defining the summand:

$$Au := (n+1-k)u_{n+1,k} - (n+1)u_{n,k} = 0, \qquad Bu := (k+1)u_{n,k+1} - (n-k)u_{n,k} = 0.$$

The aim is to derive a recurrence satisfied by S_n from these equations. This is done by first finding an equation satisfied by $u_{n,k}$ where k does not appear in the coefficients. Such an equation is given by Pascal's triangle rule $u_{n+1,k+1} = u_{n,k+1} + u_{n,k}$ which can be deduced from the above equations as $(S_k+1)A + S_n B$, where S_k (resp. S_n) denotes the shift with respect to k (resp. n). This equation is then rewritten in a form suitable for summation with respect to k:

$$(u_{n+1,k+1} - u_{n+1,k}) - (u_{n,k+1} - u_{n,k}) + u_{n+1,k} - 2u_{n,k} = 0.$$

Since the binomial coefficient $\binom{n}{k}$ is 0 when $k < 0$ or $k > n$, summing over k simply yields the desired result $S_{n+1} - 2S_n = 0$ (this is where telescoping takes place). Using the initial condition $S_0 = 1$, any solver of recurrence equations would then produce $S_n = 2^n$.

A similar example is provided by $U_n = \sum_{k=0}^{n} \binom{n}{k}^2$. The system of equations is a simple modification of the former one. Finding an equation which does not involve k in the coefficients is slightly harder. One finds

$$(n+1)u_{n+2,k+2} - (2n+3)u_{n+1,k+2} + (n+1)u_{n,k+2} - (2n+3)u_{n+1,k+1} - 2(n+1)u_{n,k+1} + u_{n,k} = 0.$$

Again, this is rewritten in a form where telescoping will take place by repeatedly expressing $v_{k+1} = (v_{k+1} - v_k) + v_k$. Summing then yields

$$(n+1)U_{n+1} - 2(2n+1)U_n = 0.$$

Again, with the initial condition $U_0 = 1$, it is easy to conclude that $U_n = \binom{2n}{n}$.

Exactly the same computation applies to definite integrals. For instance, to compute $F(x) = \int_{-\infty}^{+\infty} \exp(-xy^2)\,dy$, one starts from a system satisfied by the integrand

$$D_x + y^2 = 0, \qquad D_y + 2xy = 0,$$

where D_x denotes differentiation with respect to x (and similarly for D_y). Then we look for an equation satisfied by f without y in the coefficients. It is not difficult to find that such an equation is $(D_y^2 + 4x^2 D_x + 2x)f = 0$. Since for any value of x, $\exp(-xy^2)$ and its derivatives with respect to y tend to 0 at $\pm\infty$, integrating this equation over y yields $4x^2F'(x) + 2xF(x) = 0$. The initial condition $F(1) = \sqrt{\pi}$ leads to $F(x) = \sqrt{\pi/x}$.

2. Ore algebras

A very natural framework to describe creative telescoping is provided by a special case of skew polynomial rings called Ore algebras. These are algebras of linear operators which generalize the difference and differential operators.

Definition 1. Let $\mathbb{K}$ be a (possibly skew) field. Let $\partial_1, \ldots, \partial_r$ be defined by the following commutation rules with all the elements P in $A = \mathbb{K}(x_1, \ldots, x_p)[y_1, \ldots, y_q]$:

$$\partial_i P = \sigma_i(P)\partial_i + \delta_i(P),$$

where σ_i is a ring endomorphism of A and δ_i is an additive endomorphism which satisfies the following Leibniz rule:

$$\delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b, \qquad \forall a, b \in A.$$

Then $\mathbb{K}(x_1, \ldots, x_p)[y_1, \ldots, y_q]\langle \partial_1, \ldots, \partial_r\rangle$ is called an *Ore algebra.*

Examples of Ore operators are given in Table 1. These can be combined in an algebra where each operator acts on a different variable. For instance, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be described in $\mathbb{Q}(\alpha, \beta, x, n)\langle S_n, D_x\rangle$ by a linear differential equation and a linear recurrence.

More complicated examples arise when one of the ∂_i has a special commutation rule with several of the commutative variables. For instance, in $\mathbb{Q}(n, q, q^n)\langle S_n^{(q)}\rangle$, the q-shift operator satisfies the following commutation rule:

$$S_n^{(q)} n^i (q^n)^j = q^j (n+1)^i (q^n)^j S_n^{(q)}.$$

In this framework, creative telescoping becomes an elimination process. Given a set of operators generating an ideal of operators which vanish on the function we want to sum or integrate, the main

Operator ∂	$\sigma(a)$	$\delta(a)$	Commutation	Action of ∂
Differentiation	$a(x)$	$a'(x)$	$\partial x = x\partial + 1$	$f(x) \mapsto f'(x)$
Shift	$a(x+1)$	0	$\partial x = (x+1)\partial$	$f(x) \mapsto f(x+1)$
Difference	$a(x+1)$	$a(x+1) - a(x)$	$\partial x = (x+1)\partial + 1$	$f(x) \mapsto f(x+1) - f(x)$
q-Dilation	$a(qx)$	0	$\partial x = qx\partial$	$f(x) \mapsto f(qx)$
q-Difference	$a(qx)$	$a(qx) - a(x)$	$\partial x = qx\partial + (q-1)x$	$f(x) \mapsto f(qx) - f(x)$
q-Differentiation	$a(qx)$	$\frac{a(qx)-a(x)}{(q-1)x}$	$\partial x = qx\partial + 1$	$f(x) \mapsto \frac{f(qx)-f(x)}{(q-1)x}$
Eulerian operator	$a(x)$	$xa(x)$	$\partial x = x\partial + x$	$f(x) \mapsto xf'(x)$
e^t-Differentiation	$a(x)$	$xa(x)$	$\partial x = x\partial + x$	$f(t) \mapsto f'(t) \qquad (x = e^t)$
Mahlerian operator	$a(x^p)$	0	$\partial x = x^p\partial$	$f(x) \mapsto f(x^p) \qquad (p \geq 2)$

Table 1. Ore operators

step of creative telescoping asks for an operator in the ideal that does not involve the variable with respect to which we want to integrate or sum. It turns out that under mild conditions on the σ_i's and δ_i's, Ore algebras are Noetherian and an extension of Buchberger's algorithm can be used to compute Gröbner bases. The elimination necessary for creative telescoping can thus be performed automatically provided we have a good description of the ideal.

Given an ideal $\mathcal{I}$ and an operator ∂ of the Ore algebra $\mathcal{O} = \mathbb{K}[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_k\rangle$, let $\mathbf{x}$ be those elements of $\{x_1,\dots,x_n\}$ which commute with ∂. The first step of creative telescoping is therefore to find a basis of the ideal $\mathcal{J} = \mathcal{I} \cap \mathbb{K}[\mathbf{x}]\langle\partial_1,\dots,\partial_k\rangle$ by elimination. The elements of $\mathcal{J}$ can be written

$$\partial A + B, \tag{1}$$

where B does not involve ∂. Since this is an element of $\mathcal{I}$, it cancels whatever function f the ideal $\mathcal{I}$ was cancelling. Now assuming Af to be 0 on the "borders" of the domain, multiplying by ∂^{-1} shows that B is the result we are after (see [2] for a more rigourous description and the application to indefinite operations).

3. More examples

The computation of Gröbner bases of Ore algebras has been implemented by F. Chyzak in his *Mgfun* Maple package available at the URL `http://www-rocq.inria.fr/algo/`. We now illustrate some uses of this package.

3.1. Generating Function of the Jacobi Polynomials. The idea is first to define operators annihilating $P_n^{(\alpha,\beta)}(x)y^n$ and then to compute the sum over n by creative telescoping.

We start with two operators in D_x and S_n annihilating $P_n^{(\alpha,\beta)}(x)$ (omitted here for space reasons):

```
G:=[...,...]:
```

We then load the package and define the Ore algebra in which this computation will take place.

```
with(Mgfun):
A:=orealg(diff=[Dx,x],diff=[Dy,y],shift=[Sn,n],comm=[alpha,beta]):
```

This expresses that there are two variables with a differentiation-like commutation rule, one variable with a shift-like commutation rule and two commutative variables. From the operators annihilating $P_n^{(\alpha,\beta)}(x)$, it is easy to derive operators annihilating $P_n^{(\alpha,\beta)}(x)y^n$:

```
G:=map(primpart,map(numer,[op(subs(Sn=Sn/y,G)),y*Dy-n]),[Sn,Dx,Dy]):
```

Then we are ready for elimination: we create an appropriate term order and then compute a Gröbner basis with respect to it:

```
T:=termorder(A,lexdeg=[[n],[Sn,Dx,Dy]]):
GB:=gbasis(G,T,ratpoly(rational,[x,y,alpha,beta])):
```

We finally select those operators in this basis which do not involve n, and sum over n, which is equivalent to taking the remainder of the division by Δ_n:

```
subs(Sn=1,remove(has,GB,n)):
```

The computation has taken 17 seconds (on a Dec Alpha). After a further fast Gröbner basis computation, the result is reduced to a system of two equations, a large one of order 2 in D_y and another one linear in D_x and D_y. It is then possible to interact with a differential equation solver and, using the initial conditions, obtain the closed-form formula

$$F(x,y) = \frac{1}{R(1-y+R)^a(1+y+R)^b}, \qquad R = \sqrt{1-2xy+y^2}.$$

3.2. q**-Dixon identity.** The aim is to show that

$$\sum_k (-1)^k q^{\frac{k(3k+1)}{2}} \binom{a+b}{a+k}_q \binom{b+c}{b+k}_q \binom{a+c}{c+k}_q = \binom{a+b+c}{a,b,c}_q. \tag{2}$$

The algebra is $\mathbb{Q}(q, q^a, q^b, q^c, q^k)\langle S_a, S_b, S_c, S_k\rangle$ which has only q-shift operators:

```
A:=orealg(comm=[q],qshift=[Sa,qa,q],qshift=[Sb,qb,q],
          qshift=[Sc,qc,q],qshift=[Sk,qk,q]):
```

The operators defining the summand are all of order 1 and can be obtained in *Mgfun* by

```
G:=subs([q^a=qa,q^b=qb,q^c=qc,q^k=qk], hypergeomtoholon((-1)^k*q^(k*(3*k+1)/2)
      *qbinomial(a+b,a+k)*qbinomial(a+c,c+k)*qbinomial(b+c,b+k),A)):
```

Then we eliminate q^k and proceed with the telescoping:

```
T:=termorder(A,lexdeg=[[qk],[Sa,Sb,Sc,Sk]]):
GB:=gbasis(G,T,ratpoly(rational,[q,a,b,c,qa,qb,qc])):
CT:=subs(Sk=1,remove(has,GB,[k,qk])):
```

This yields a system of operators symmetrical in a, b, c. Using one more Gröbner basis computation, one obtains an operator involving only S_a. By symmetry similar operators in S_b and S_c can be found. Then checking that the right-hand side of (2) satisfies these equations and that sufficiently many initial condition coincide proves the identity. It is also possible to use Abramov and Petkovšek's q-version of Petkovšek's algorithm to find the right-hand side.

4. Takayama's algorithm

The computation of A and B in (1) is slightly more than what is strictly necessary. Actually we only need to compute B. N. Takayama gave an algorithm for doing so in the Weyl algebra, and this algorithm generalizes to Ore algebras.

The idea is that it is possible to throw away all the right multiples of ∂ during the computation as long as we know they will only be multiplied by polynomials which commute with ∂ during later computations (so that they will remain right multiples of ∂). This is done by working in increasingly large modules where multiplication by the x_i's which do not commute with ∂ is forbidden. The operator ∂ can then easily be eliminated in a preprocessing phase.

This results in an algorithm which is generally faster than the general one, but which is only guaranteed to terminate when there is an element free of the undesirable variables in the ideal.

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