This paper appeared in *Mathematics Magazine* Vol 67 (1994) pp294–300. The pagination of this copy follows the original.

The Importance of Being Continuous

D.J. Jeffrey
The University of Western Ontario, London, Ontario, Canada N6A 5B7

1. Introduction. Every student of calculus learns its fundamental theorem, usually stated as follows [**1**]. *If f is continuous on $[a,b]$, then the function defined by*

$$g(x) = \int_a^x f(t)dt \tag{1}$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$. Once students have learned this theorem, they proceed to learn a variety of ways to obtain closed-form expressions for the integrals of given integrands. These ways include memorizing a table of simple integrals, applying substitutions, learning to consult tables of integrals such as the CRC handbook [**2**] or Gradshteyn and Ryzhik [**3**], and, increasingly, asking a computer algebra system, such as *Derive*, *Maple* or *Mathematica*.

The first integrations that a student sees performed always result in closed-form expressions that are correct for the whole domain of the integrand. For example, when a textbook demonstrates how to obtain the result

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

using the substitution $x = \sin\theta$, there is usually no accompanying statement that the integrand is continuous on the interval $(-1,1)$ and that $\arcsin x$ is the correct integral over that whole interval. Mavbe it is obvious. By working through problem sets, students accumulate examples of integrals that reduce to expressions which are valid everywhere that the integrand is continuous. It is inevitable that students then jump, perhaps unconsciously, to the conclusion that any closed-form result obtained by a substitution, or found in a book, must be valid on an interval equal to the domain of the integrand. This paper considers some cases in which that conclusion is false.

Consider the function $f = \sqrt{1-\cos x}$. Figure 1 shows that f is continuous for all x, and therefore the fundamental theorem says there must exist a function g, also continuous for all x, which is its integral. Using the substitution $u = \cot\frac{1}{2}x$, we can derive the equation

$$\int \sqrt{1-\cos x}\, dx = -2\sqrt{1-\cos x}\cot\tfrac{1}{2}x\ . \tag{2}$$

This equation is returned by *Mathematica*, for example. The right-hand side of (2) is discontinuous at even multiples of π, as figure 1 shows, and so although it was obtained by a standard procedure, it is not the integral g that the theorem says exists.

There are two attitudes we might take towards an equation such as (2). On the one hand, we could say that since (2) is correct only on the interval $(0,2\pi)$, the fault is one of notation. Technically, it is always incorrect to quote a formula without specifying the interval on which it applies, and this is a case in which the common practice of leaving the interval unspecified is particularly misleading. With

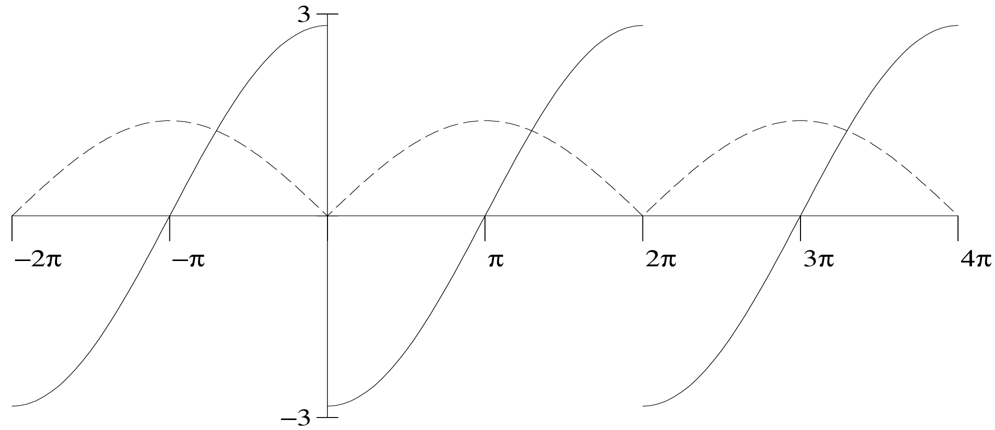

Fig 1. The integrand and integral in (2). − − − , the integrand. ——, the discontinuous integral.

---

this attitude, the problem becomes one of determining the interval of validity of a given closed-form expression and ensuring that the interval is properly displayed. On the other hand, we could say that since the fundamental theorem tells us that there does exist an integral of f that is continuous everywhere on the real line, our object should be to capture it in a closed-form expression. With this attitude, the problem becomes one of modifying expressions like (2) to enlarge their intervals of validity, and this is the view we take here.

We can contrast (2) with the equation

$$\int \frac{dx}{x^2} = -\frac{1}{x} \ . \tag{3}$$

This equation is not valid on intervals containing the origin, and again textbooks are unlikely to say this explicitly. The difference, though, is that we cannot do better, because the discontinuity in $1/x$ is the result of $1/x^2$ violating the conditions required by the fundamental theorem at the origin. Discontinuities such as those seen in (2) should be regarded as spurious, and therefore subject to further investigation, while discontinuities or singularities such as the one in (3) are genuine and must be accepted.

This is not a paper on computer algebra systems, but the perspective they add to the problem is an interesting one, and should be taken into account. To begin with, the spread of these systems is allowing increasing numbers of students to range much more widely in calculus than before, making it more likely that they will discover the difficulties discussed here for themselves. In addition, experienced mathematicians use these systems and demand the maximum in convenience from them, which means the designers must automate tasks that previously were left unaddressed, such as allowing for discontinuities. Finally, at present no algebra system has a good method for informing the user that a result it has obtained is valid only on a restricted interval, and so it is important to return results that are valid on as wide an interval as possible.

In the following sections we shall look at various integration problems in which expressions arise that contain spurious discontinuities, in the sense defined above. As sources of discontinuous expressions, we shall consider integral tables, substitutions and special types of integrands. It is important to realize that tables of integrals and calculus textbooks harbour an astonishingly large number of expressions with spurious discontinuities, and the fact that these entries have successfully maintained their population for over 100 years indicates clearly that they have no natural predators, even though they are one species that humans should be encouraged to endanger.

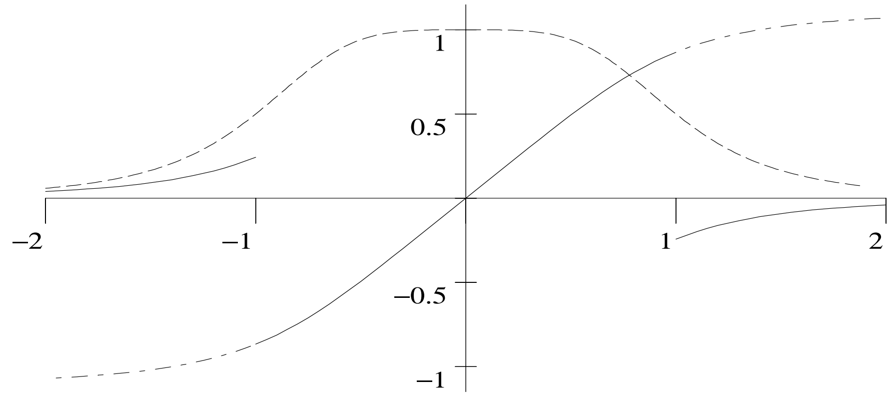

Figure 2. The functions in equations (4) and (5). − − − , the integrand. ——, the discontinuous integral (4). - · · · -, the continuous integral (5).

---

2. Tables of integrals. We discuss this source of discontinuous integrals using a specific example, an entry appearing in two of the most popular tables of integrals, Gradshteyn and Ryshik [**3**] where it is formula 2.132.1, and the CRC Press Handbook [**2**] where it is formula 77.

$$\int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}} \ln \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}} \arctan \frac{x\sqrt{2}}{1-x^2} \ . \tag{4}$$

The left-hand side of this equation is continuous for all x, but the right-hand side contains discontinuities at $x = \pm 1$. There is nothing printed in the tables to warn the reader of the discontinuities or, equivalently, to advise the reader of the intervals on which the formula is valid. To make matters worse, the editors of these tables have overlooked an alternative to (4) which is free of discontinuities.

$$\int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}} \ln \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}} \arctan(x\sqrt{2}+1) + \frac{1}{2\sqrt{2}} \arctan(x\sqrt{2}-1) \ . \tag{5}$$

Figure 2 shows plots of (4) and (5) on the same axes.

Equation (5) shows that the discontinuities in (4) are unnecessary, and even misleading, because they masquerade as properties of the integrand, whereas they are really artefacts of the process that derived the expression. It is likely that (4) was originally derived by obtaining (5) first and then making incorrect use of the formula [**3**]

$$\arctan x + \arctan y = \begin{cases} \arctan \dfrac{x+y}{1-xy} \ , & \text{for } xy < 1, \\ \arctan \dfrac{x+y}{1-xy} + \pi \ , & \text{for } xy > 1 \text{ and } x > 0, \\ \arctan \dfrac{x+y}{1-xy} - \pi \ , & \text{for } xy > 1 \text{ and } x < 0. \end{cases}$$

Probably, the first line of this formula was taken to apply for all x and y, a misconception easily carried away from a casual reading of Abramowitz and Stegun [**4**]. By comparing (4) and (5), we can see that there is a temptation to prefer (4) because of its compactness, but the discontinuities it introduces negate this advantage.

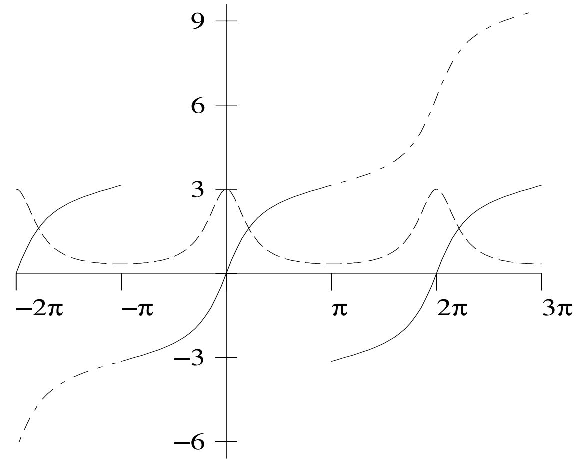

Figure 3. The functions in equations (6) and (7). − − − , the integrand. ——, the discontinuous integral (6). - · · · -, the continuous integral (7).

A search through any extensive table of integrals will find entries in which there are terms of the form $\arctan(P/Q)$, where P and Q are polynomials. If Q has roots within the range of integration, then the expression will contain discontinuities. Tables of integrals should correct such entries. A form for an integral that is free of spurious discontinuities must be counted as superior to one that is not, and it is surely shoddy editing to print the inferior form. Also, there is the question of efficiency. If the user must check every entry in the tables for continuity, the use of the tables becomes less efficient.

3. Discontinuity from substitution. The most important substitution that leads to spurious discontinuities is the Weierstrass, or $\tan \frac{1}{2}x$, substitution. For example, the function $3/(5 - 4\cos x)$ is continuous and positive for all real x, and so its integral should be continuous and monotonically increasing. By letting $u = \tan \frac{1}{2}x$, we obtain

$$\int \frac{3\,dx}{5 - 4\cos x} = \int \frac{6\,du}{1 + 9u^2} = 2\arctan(3\tan \tfrac{1}{2}x) \ . \tag{6}$$

The final expression in (6) is discontinuous at odd multiples of π, as can be seen in figure 3. The unsatisfactory nature of (6) has rarely been noted. The only published commentary on this class of integrals that acknowledges that special treatment is required is the introduction to the CRC tables [**2**] where it is stated that the 'correct branch' of the inverse tangent must be used when applying the formula.

A new method for evaluating integrals such as (6) that always yields continuous expressions has been developed [**5**]. When applied to the integral above, it gives

$$\int \frac{3\,dx}{5 - 4\cos x} = 2\arctan(3\tan \tfrac{1}{2}x) + 2\pi \left\lfloor \frac{x + \pi}{2\pi} \right\rfloor \ , \tag{7}$$

where $\lfloor\ \rfloor$ is the floor function. An interesting facet of this result is that each term separately is discontinuous at odd multiples of π, but the algorithm arranges things so that together they make a continuous function. Figure 3 shows plots of the functions appearing in (6) and (7). We can express the floor function in terms of an inverse tangent as

$$2\pi\lfloor (x + \pi)/2\pi \rfloor = x - 2\arctan(\tan \tfrac{1}{2}x) \ ,$$

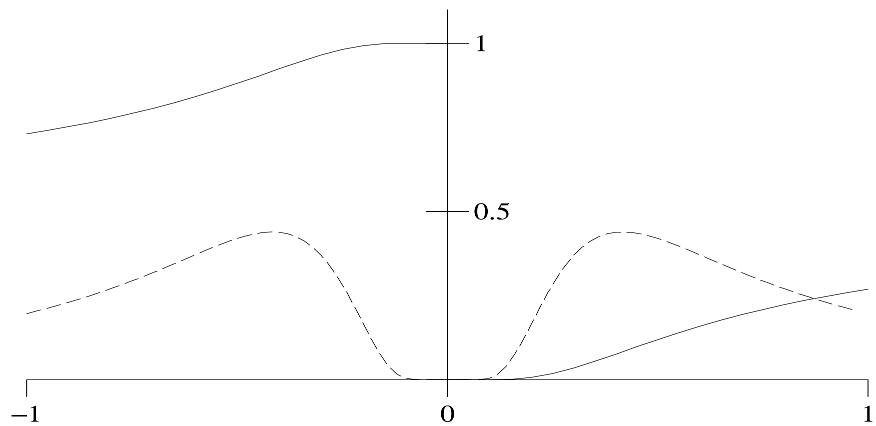

Figure 4. The functions in equation (10). – – – , the integrand. ——, the discontinuous integral.

and substituting this into (7) and combining inverse tangents gives the following compact variation on (7):

$$\int \frac{3\,dx}{5-4\cos x} = x + 2\arctan\frac{\sin x}{2-\cos x}\ . \tag{8}$$

A generalization of this expression that is suitable for integral tables is as follows. If

$$p^2 > q^2 + r^2\ ,$$

then

$$\int \frac{dx}{p+q\cos x + r\sin x} = \frac{x}{\Delta} + \frac{2}{\Delta}\arctan\frac{r\cos x - q\sin x}{p+q\cos x + r\sin x + \Delta}\ , \tag{9}$$

where $\Delta = \operatorname{sgn}(p)\sqrt{p^2-q^2-r^2}$. The right-hand side is continuous for all possible values of the parameters, and should be used in place of all similar formulae, wherever found.

Any substitution $x = u(s)$ can lead to an antiderivative with a spurious discontinuity if $u(s)$ contains a singularity. For example, since the substitution $x = 1/s$ is singular at the origin, results obtained using it should be checked for continuity there. The integrand in (10) is continuous everywhere, having a removable singularity at the origin, but the substitution $1/s$ gives a discontinuous integral.

$$\int \frac{e^{1/x}}{(1+e^{1/x})^2}\frac{dx}{x^2} = \int \frac{-e^s}{(1+e^s)^2}ds = \frac{1}{1+e^{1/x}}\ . \tag{10}$$

The integrand and integral are plotted in figure 4. We can remove the jump by using the signum function. The jump in the function at the origin is -1, and so a continuous expression for the integral is

$$\int \frac{e^{1/x}}{(1+e^{1/x})^2}\frac{dx}{x^2} = \frac{1}{1+e^{1/x}} + \tfrac{1}{2}\operatorname{sgn} x\ .$$

4. Piecewise continuous functions. Introductory calculus books and integral tables do not address the problems of integrating piecewise continuous functions, even though the step function, and similar functions, are very useful in applications and are used freely in physics and engineering books. Recently, Botsko [**6**] has stated and proved a generalization of the fundamental theorem that requires only that f be integrable, and not necessarily continuous, and hence covers piecewise continuous functions. The central topic of this paper, namely the importance of ensuring that integrals are continuous, is an important factor in the development of correct rules for integrating piecewise-defined functions.

The Heaviside step function is defined by

$$H(x) = \begin{cases} 1, & \text{for } x \geq 0 \ ; \\ 0, & \text{for } x < 0 \ . \end{cases}$$

As an example of how we want to work with this function, consider the following differential equation coming from elementary beam theory. The bending moment $M(x)$ in a beam extending from $x = 0$ to $x = l$ and supporting point loads P_a and P_b at $x = a$ and $x = b$ is given by the equation

$$\frac{dM}{dx} = \begin{cases} K, & \text{for } 0 \leq x \leq a \ ; \\ K + P_a, & \text{for } a \leq x \leq b \ ; \\ K + P_a + P_b, & \text{for } b \leq x \leq l \ . \end{cases}$$

where K is an unknown constant. The equation is to be solved subject to boundary conditions representing free ends, to wit, $M(0) = M(l) = 0$. Most students would solve this by integrating each line separately to obtain

$$M = \begin{cases} Kx + A_1, & 0 \leq x \leq a \ ; \\ (K + P_a)x + A_2, & a \leq x \leq b \ ; \\ (K + P_a + P_b)x + A_3, & b \leq x \leq l \ . \end{cases}$$

Matching the solutions at $x = a$ and at $x = b$ gives

$$A_1 = A_2 + P_a a \qquad \text{and} \qquad A_2 = A_3 + P_b b \ .$$

The boundary conditions now give $A_1 = 0$ and $K = P_a(a - l)/l + P_b(b - l)/l$. The problem is not finished, however, because we must integrate twice more the same way to obtain the deflection.

A much quicker way to proceed is to write

$$\frac{dM}{dx} = K + P_a H(x - a) + P_b H(x - b) \ ,$$

and develop rules for integrating H. At first sight, it might seem that the rule is very simple, namely

$$\int f(x)H(x - a)dx = H(x - a) \int f(x)dx \ .$$

For most functions f, however, the right-hand side will violate our principle that the integral must be continuous. The better form is

$$\int f(x)H(x - a)dx = H(x - a) \int_a^x f(t)dt \ . \tag{11}$$

The right-hand side is now continuous everywhere that the integral of f is. Applying this to the differential equation above, we get

$$M = (x - a)P_a H(x - a) + (x - b)P_b H(x - b) + Kx + A \ ,$$

and the boundary conditions give the same solution as before. The deflection can now be easily obtained by integrating twice more to find

$$y = \tfrac{1}{6}(x-a)^3 P_a H(x-a) + \tfrac{1}{6}(x-b)^3 P_b H(x-b) + \tfrac{1}{6}Kx^3 + Bx + C \ .$$

The boundary conditions $y(0) = y(l) = 0$ give us B and C. This method of solution is more convenient than the first one for people working by hand, and vastly more convenient for those using an algebra system.

5. Conclusions. In calculus textbooks, it is popular to include a section on the use of integral tables. In view of the results in section 2, the textbooks should warn students that any expression extracted from a table might contain a spurious discontinuity. With equal force, we should require the editors of handbooks to check their tables thoroughly. In effect, an entry in a table should not be considered correct unless it is continuous on as wide an interval as possible. The alternative would be to note the interval upon which the integral is valid, without attempting to broaden it, but this would be less useful to the reader. Similar comments can be applied to computer algebra systems.

The Weierstrass substitution discussed in section 3 was once a standard topic in calculus textbooks, albeit an advanced topic. It appears less frequently now, but if it is treated, I think that an analysis of the discontinuity and its correct handling must be included. The material of section 4 comes from my experience of watching students tackle problems such as the one described, and from implementing the solution on algebra systems.

This paper springs directly from discussions with David Stoutemyer and Al Rich, the developers of *Derive*, a computer algebra program. Almost all of the items discussed here have been implemented in *Derive*, and I am grateful to its developers for their interest. I am also grateful to the developers of *Maple* for stimulating discussions, and for adopting some of the ideas above.

References.

1. J. Stewart, *Calculus, 2nd Ed.* Brooks/Cole, Pacific Grove, CA, 1991.
2. W.H. Beyer, *Handbook of Mathematical Sciences, 6th Ed.* CRC Press, Boca Raton, FL, 1987.
3. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products,* Academic Press, San Diego, CA, 1979.
4. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions,* Dover, New York, NY, 1965.
5. D.J. Jeffrey and A.D. Rich, The evaluation of trigonometric integrals avoiding spurious discontinuities, *ACM Trans. Math. Software*, submitted.
6. M.W. Botsko, A fundamental theorem of calculus that applies to all Riemann integrable functions, *this* MAGAZINE , **64** (1991), 347–348.