
Necessary and Sufficient Conditions for Differentiating Under the Integral Sign

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1. INTRODUCTION. When we have an integral that depends on a parameter, say $F(x) = \int_a^b f(x, y) dy$, it is often important to know when F is differentiable and when $F'(x) = \int_a^b f_1(x, y) dy$. A sufficient condition for differentiating under the integral sign is that $\int_a^b f_1(x, y) dy$ converges uniformly; see [6, p. 260]. When we have absolute convergence, the condition $|f_1(x, y)| \leq g(y)$ with $\int_a^b g(y) dy < \infty$ suffices (Weierstrass M-test and Lebesgue Dominated Convergence). If we use the Henstock integral, then it is not difficult to give necessary and sufficient conditions for differentiating under the integral sign. The conditions depend on being able to integrate every derivative.

If $g : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) it is not always the case that g' is Riemann or Lebesgue integrable over $[a, b]$. However, the Henstock integral integrates all derivatives and thus leads to the most complete version of the Fundamental Theorem of Calculus. The Henstock integral's definition in terms of Riemann sums is only slightly more complicated than for the Riemann integral (simpler than the improper Riemann integral), yet it includes the Riemann, improper Riemann, and Lebesgue integrals as special cases. Using the very strong version of the Fundamental Theorem we can formulate necessary and sufficient conditions for differentiating under the integral sign.

2. AN INTRODUCTION TO THE HENSTOCK INTEGRAL. Here we lay out the facts about Henstock integration that we need. There are now quite a number of works that deal with this integral; two good ones to start with are [1] and [3].

Let $f : [-\infty, \infty] \rightarrow (\infty, \infty)$. A *gauge* is a mapping γ from $[-\infty, \infty]$ to the open intervals in $[-\infty, \infty]$. By *open interval* we mean (a, b) , $[-\infty, b)$, $(a, \infty]$, or $[-\infty, \infty]$ for all $-\infty \leq a < b \leq \infty$ (the two-point compactification of the real line). The defining property of the gauge is that for all $x \in [-\infty, \infty]$, $\gamma(x)$ is an open interval containing x . A *tagged partition* of $[-\infty, \infty]$ is a finite set of pairs $\mathcal{P} = \{(z_i, I_i)\}_{i=1}^N$, where each I_i is a nondegenerate closed interval in $[-\infty, \infty]$ and $z_i \in I_i$. The points $z_i \in [-\infty, \infty]$ are called *tags* and need not be distinct. The intervals $\{I_i\}_{i=1}^N$ form a *partition*: For $i \neq j$, $I_i \cap I_j$ is empty or a singleton and $\cup_{i=1}^N I_i = [-\infty, \infty]$. We say \mathcal{P} is γ -*fine* if $I_i \subset \gamma(z_i)$ for all $1 \leq i \leq N$. Let $|I|$ denote the length of an interval with $|I| = 0$ for an unbounded interval. Then, f is *Henstock integrable*, and we write $\int_{-\infty}^{\infty} f = A$, if there is a real number A such that for all $\epsilon > 0$ there is a gauge function γ such that if $\mathcal{P} = \{(z_i, I_i)\}_{i=1}^N$ is any γ -fine tagged partition of $[-\infty, \infty]$ then

$$\left| \sum_{i=1}^N f(z_i) |I_i| - A \right| < \epsilon.$$

Note that N is not fixed and the partitions can have any finite number of terms. We can integrate over an interval $[a, b] \subset [-\infty, \infty]$ by multiplying the integrand with the characteristic function $\chi_{[a,b]}$.

The more dramatically a function changes near a point z , the smaller $\gamma(z)$ becomes. With the Riemann integral the intervals are made uniformly small. Here they are locally small. A function is *Riemann integrable* on a finite interval if and only if the gauge can be taken to assign intervals of constant length. It is not too surprising that the Henstock integral includes the Riemann integral. What is not so obvious is that the Lebesgue integral is also included. And, the Henstock integral can integrate functions that are neither Riemann nor Lebesgue integrable. An example is the function $f = g'$ where $g(x) = x^2 \sin(1/x^3)$ for $x \neq 0$ and $g(0) = 0$; the origin is the only point of nonabsolute summability. See [5, p. 148] for a function that is Henstock integrable but whose points of nonabsolute summability have positive measure. A key feature of the Henstock integral is that it is nonabsolute: an integrable function need not have an integrable absolute value.

The convention $|I| = 0$ for an unbounded interval performs essentially the same truncation that is done with improper Riemann integrals and the Cauchy extension of Lebesgue integrals. A consequence of this is that there are no improper Henstock integrals. This fact is captured in the following theorem, which is proved for finite intervals in [3].

Theorem 1. *Let f be a real-valued function on $[a, b] \subseteq [-\infty, \infty]$. Then $\int_a^b f$ exists and equals $A \in \mathbb{R}$ if and only if f is integrable on each subinterval $[a, x] \subset [a, b]$ and $\lim_{x \rightarrow b^-} \int_a^x f$ exists and equals A .*

Lebesgue integrals can be characterised by the fact that the indefinite integral $F(x) = \int_a^x f$ is absolutely continuous. A similar characterisation is possible with the Henstock integral. We need three definitions. Let $F : [a, b] \rightarrow \mathbb{R}$. We say F is *absolutely continuous (AC)* on a set $E \subseteq [a, b]$ if for each $\epsilon > 0$ there is some $\delta > 0$ such that $\sum_{i=1}^N |F(x_i) - F(y_i)| < \epsilon$ for all finite sets of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^N$ with endpoints in E and $\sum_{i=1}^N (y_i - x_i) < \delta$. We say that F is *absolutely continuous in the restricted sense (AC_{*})* if instead we have $\sum_{i=1}^N \sup_{x, y \in [x_i, y_i]} |F(x) - F(y)| < \epsilon$ under the same conditions as with AC. And, F is said to be *generalised absolutely continuous in the restricted sense (ACG_{*})* if F is continuous and E is the countable union of sets on each of which F is AC_{*}. Two useful properties are that among continuous functions, the ACG_{*} functions are properly contained in the class of functions that are differentiable almost everywhere and they properly contain the class of functions that are differentiable nearly everywhere (differentiable except perhaps on a countable set). A function f is *Henstock integrable* if and only if there is an ACG_{*} function F with $F' = f$ almost everywhere. In this case $F(x) - F(a) = \int_a^x f$. For an unbounded interval such as $[0, \infty]$, continuity of f at ∞ is obtained by demanding that $\lim_{x \rightarrow \infty} F(x)$ exists.

We have given an example that shows that not all derivatives are Lebesgue integrable. However, all derivatives are Henstock integrable. This leads to a very strong version of the Fundamental Theorem of Calculus. A proof can be pieced together from results in [3].

Theorem 2. (Fundamental Theorem of Calculus)

I *Let $f : [a, b] \rightarrow \mathbb{R}$. Then $\int_a^b f$ exists and $F(x) = \int_a^x f$ for all $x \in [a, b]$ if and only if F is ACG_{*} on $[a, b]$, $F(a) = 0$, and $F' = f$ almost everywhere on (a, b) . If $\int_a^b f$ exists and f is continuous at $x \in (a, b)$ then $\frac{d}{dx} \int_a^x f = f(x)$.*

II Let $F : [a, b] \rightarrow \mathbb{R}$. Then F is ACG_* if and only if F' exists almost everywhere on (a, b) , F' is Henstock integrable on $[a, b]$, and $\int_a^x F' = F(x) - F(a)$ for all $x \in [a, b]$.

Here is a useful sufficient condition for integrability of the derivative:

Corollary 3. Let $F : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable nearly everywhere on (a, b) . Then F' is Henstock integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$ for all $x \in [a, b]$.

The improvement over the Riemann and Lebesgue cases is that we need not assume the integrability of F' . Integration and differentiation are now inverse operations. To make this explicit, let A be the vector space of Henstock integrable functions on $[a, b] \subseteq [-\infty, \infty]$, identified almost everywhere. Let B be the vector space of ACG_* functions vanishing at a . Let \int be the integral operator defined by $\int[f](x) = \int_a^x f$ for $f \in A$. Let D be the differential operator defined by $D[f](x) = f'(x)$ for $f \in B$. The Fundamental Theorem then says that $D \circ \int = I_A$ and $\int \circ D = I_B$.

3. DIFFERENTIATION UNDER THE INTEGRAL SIGN.

Theorem 4. Let $f : [\alpha, \beta] \times [a, b] \rightarrow \mathbb{R}$. Suppose that $f(\cdot, y)$ is ACG_* on $[\alpha, \beta]$ for almost all $y \in (a, b)$. Then $F := \int_a^b f(\cdot, y) dy$ is ACG_* on $[\alpha, \beta]$ and $F'(x) = \int_a^b f_1(x, y) dy$ for almost all $x \in (\alpha, \beta)$ if and only if

$$\int_{x=s}^t \int_{y=a}^b f_1(x, y) dy dx = \int_{y=a}^b \int_{x=s}^t f_1(x, y) dx dy \quad \text{for all } [s, t] \subseteq [\alpha, \beta]. \quad (1)$$

Proof. Suppose F is ACG_* and $\frac{\partial}{\partial x} \int_a^b f(x, y) dy = \int_a^b f_1(x, y) dy$. Let $[s, t] \subseteq [\alpha, \beta]$. By the second part of the Fundamental Theorem, applied first to F and then to $f(\cdot, y)$,

$$\begin{aligned} \int_{x=s}^t \int_{y=a}^b f_1(x, y) dy dx &= F(t) - F(s) = \int_{y=a}^b [f(t, y) - f(s, y)] dy \\ &= \int_{y=a}^b \int_{x=s}^t f_1(x, y) dx dy. \end{aligned} \quad (2)$$

Now assume (1). Let $x \in (\alpha, \beta)$ and let $h \in \mathbb{R}$ be such that $x + h \in (\alpha, \beta)$. Then, applying the second part of the Fundamental Theorem to $f(\cdot, y)$ gives

$$\begin{aligned} \int_{x'=x}^{x+h} \int_{y=a}^b f_1(x', y) dy dx' &= \int_{y=a}^b \int_{x'=x}^{x+h} f_1(x', y) dx' dy = \int_{y=a}^b [f(x+h, y) - f(x, y)] dy \\ &= \int_{y=a}^b f(x+h, y) dy - \int_{y=a}^b f(x, y) dy. \end{aligned} \quad (3)$$

And,

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} [F(x+h) - F(x)] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{x'=x}^{x+h} \int_{y=a}^b f_1(x', y) dy dx' \\
 &= \int_a^b f_1(x, y) dy \quad \text{for almost all } x \in (\alpha, \beta).
 \end{aligned}$$

The last line comes from the first part of the Fundamental Theorem. Repeating the argument in (2) shows that $\int_\alpha^x F' = F(x) - F(\alpha)$ for all $x \in [\alpha, \beta]$. Hence, F is ACG_* on $[\alpha, \beta]$. ■

The theorem holds for $-\infty \leq \alpha < \beta \leq \infty$ and $-\infty \leq a < b \leq \infty$. Partial versions of the theorem are given in [4, p. 357] for the Lebesgue integral and in [2, p. 63] for the wide Denjoy integral, which includes the Henstock integral.

From the proof of the theorem it is clear that only the linearity of the integral over $y \in [a, b]$ ((2) and (3)) comes into play with this variable. Hence, we have the following generalisation.

Corollary 5. *Let S be some set and suppose $f : [\alpha, \beta] \times S \rightarrow \mathbb{R}$. Let $f(\cdot, y)$ be ACG_* on $[\alpha, \beta]$ for all $y \in S$. Let T be the real-valued functions on S and let \mathcal{L} be a linear functional defined on a subspace of T . Define $F : [\alpha, \beta] \rightarrow \mathbb{R}$ by $F(x) = \mathcal{L}[f(x, \cdot)]$. Then F is ACG_* on $[\alpha, \beta]$ and $F'(x) = \mathcal{L}[f_1(x, \cdot)]$ for almost all $x \in (\alpha, \beta)$ if and only if*

$$\int_s^t \mathcal{L}[f_1(x, \cdot)] dx = \mathcal{L} \int_s^t f_1(x, \cdot) dx \quad \text{for all } [s, t] \subseteq [\alpha, \beta].$$

If $f(x, y) = \int_\alpha^x g(x', y) dx'$ then $f(\cdot, y)$ is automatically ACG_* . This gives necessary and sufficient conditions for interchanging iterated integrals.

Corollary 6. *Let $g : [\alpha, \beta] \times [a, b] \rightarrow \mathbb{R}$. Suppose that $g(\cdot, y)$ is integrable over $[\alpha, \beta]$ for almost all $y \in (a, b)$. Define $G(x) = \int_a^b \int_\alpha^x g(x', y) dx' dy$. Then G is ACG_* on $[\alpha, \beta]$ and $G'(x) = \int_a^b g(x, y) dy$ for almost all $x \in (\alpha, \beta)$ if and only if*

$$\int_{x=s}^t \int_{y=a}^b g(x, y) dy dx = \int_{y=a}^b \int_{x=s}^t g(x, y) dx dy \quad \text{for all } [s, t] \subseteq [\alpha, \beta]. \quad (4)$$

Combining Corollaries 5 and 6 gives necessary and sufficient conditions for interchanging summation and integration.

Corollary 7. *Let $g : [\alpha, \beta] \times \mathbb{N} \rightarrow \mathbb{R}$ and write $g_n(x) = g(x, n)$ for $x \in [\alpha, \beta]$ and $n \in \mathbb{N}$. Suppose that g_n is integrable over $[\alpha, \beta]$ for each $n \in \mathbb{N}$. Define $G(x) = \sum_{n=1}^\infty \int_\alpha^x g_n(x') dx'$. Then G is ACG_* on $[\alpha, \beta]$ and $G'(x) = \sum_{n=1}^\infty g_n(x)$ for almost all $x \in (\alpha, \beta)$ if and only if*

$$\int_{x=s}^t \sum_{n=1}^{\infty} g_n(x) dx = \sum_{n=1}^{\infty} \int_{x=s}^t g_n(x) dx \quad \text{for all } [s, t] \subseteq [\alpha, \beta]. \quad (5)$$

The Fundamental Theorem and its corollary yield conditions sufficient to allow differentiation under the integral.

Corollary 8. Let $f: [\alpha, \beta] \times [a, b] \rightarrow \mathbb{R}$.

- i) Suppose that $f(\cdot, y)$ is continuous on $[\alpha, \beta]$ for almost all $y \in (a, b)$ and is differentiable nearly everywhere in (α, β) for almost all $y \in (a, b)$. If (1) holds then $F'(x) = \int_a^b f_1(x, y) dy$ for almost all $x \in (\alpha, \beta)$.
- ii) Suppose that $f(\cdot, y)$ is ACG_* on $[\alpha, \beta]$ for almost all $y \in (a, b)$ and that $\int_a^b f_1(\cdot, y) dy$ is continuous on $[\alpha, \beta]$. If (1) holds then $F'(x) = \int_a^b f_1(x, y) dy$ for all $x \in (\alpha, \beta)$.

Here is an example of what can go wrong when one differentiates under the integral sign without justification. In 1815 Cauchy obtained the convergent integrals

$$\int_{x=0}^{\infty} \left\{ \begin{array}{l} \sin(x^2) \\ \cos(x^2) \end{array} \right\} \cos(sx) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left[\cos\left(\frac{s^2}{4}\right) \mp \sin\left(\frac{s^2}{4}\right) \right].$$

He then differentiated under the integral sign with respect to s and obtained the two divergent integrals

$$\int_{x=0}^{\infty} x \left\{ \begin{array}{l} \sin(x^2) \\ \cos(x^2) \end{array} \right\} \sin(sx) dx = \frac{s}{4} \sqrt{\frac{\pi}{2}} \left[\sin\left(\frac{s^2}{4}\right) \pm \cos\left(\frac{s^2}{4}\right) \right].$$

These divergent integrals have been reproduced ever since and still appear in standard tables today, listed as converging. This story was told in [7].

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