Rapidly Growing Fourier Integrals

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1. THE RIEMANN–LEBESGUE LEMMA. In its usual form, the Riemann–Lebesgue Lemma reads as follows: If $f \in L^1$ and $\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$ is its Fourier transform, then $\hat{f}(s)$ exists and is finite for each $s \in \mathbb{R}$ and $\hat{f}(s) \to 0$ as $|s| \to \infty$ ($s \in \mathbb{R}$). This result encompasses Fourier sine and cosine transforms as well as Fourier series coefficients for functions periodic on finite intervals. When the integral is allowed to converge conditionally, the asserted asymptotic behaviour can fail dramatically. In fact, we show that for each sequence $a_n \uparrow \infty$ we can find a continuous function f such that $\hat{f}(s)$ exists for each $s \in \mathbb{R}$ and $\hat{f}(n) \geq a_n$ for all integers $n \geq 1$. We also work out the asymptotics of a class of Fourier integrals that can have arbitrarily large polynomial growth. Our main tool is the principle of stationary phase. The conditionally convergent integrals we consider in this paper can be thought of as Henstock integrals [**1**] or as improper Riemann integrals.

Two examples of conditionally convergent Fourier transforms that do not tend to zero at infinity can be obtained from [**3**, 3.691]:

$$\int_{x=0}^{\infty} \begin{Bmatrix} \sin(ax^2) \\ \cos(ax^2) \end{Bmatrix} \cos(sx)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left[\cos\left(\frac{s^2}{4a}\right) \mp \sin\left(\frac{s^2}{4a}\right)\right] \tag{1}$$

and

$$\int_{x=0}^{\infty} \begin{Bmatrix} \sin(ax^2) \\ \cos(ax^2) \end{Bmatrix} \sin(sx)\,dx = \sqrt{\frac{\pi}{2a}}\left[\begin{Bmatrix} \cos[s^2/(4a)] \\ \sin[s^2/(4a)] \end{Bmatrix} C\left(\frac{s^2}{4a}\right) \pm \begin{Bmatrix} \sin[s^2/(4a)] \\ \cos[s^2/(4a)] \end{Bmatrix} S\left(\frac{s^2}{4a}\right)\right]. \tag{2}$$

Here, $a > 0$ and

$$C(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \cos t\,\frac{dt}{\sqrt{t}} \quad \text{and} \quad S(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \sin t\,\frac{dt}{\sqrt{t}} \tag{3}$$

are the Fresnel integrals. Using (1) and (2) we have the Fourier transforms of $x \mapsto \sin(ax^2)$ and $x \mapsto \cos(ax^2)$. Both of these transforms oscillate rapidly at infinity with amplitude that is asymptotically constant. Note that $C(\infty) + iS(\infty) = e^{i\pi/4}/\sqrt{2}$. The values of all of these integrals are consequences of the formula $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

On a finite interval $[a, b]$ the Riemann–Lebesgue Lemma for conditionally convergent integrals takes the following form: Suppose $\int_a^b f$ exists. Then its Fourier transform, $\hat{f}(s) = \int_a^b e^{isx} f(x)\,dx$, exists for all $s \in \mathbb{R}$ since for each $s \in \mathbb{R}$, the exponential function in the integrand is of bounded variation on the finite interval $[a, b]$. Let $F(x) = \int_a^x f(t)\,dt$ and integrate by parts:

$$\hat{f}(s) = \int_a^b e^{isx} f(x)\,dx = e^{isb}F(b) - is\int_a^b e^{isx}F(x)\,dx.$$

Since $F \in L^1$ the Riemann–Lebesgue Lemma gives $\hat{f}(s) = o(s)$ as $|s| \to \infty$ in $\mathbb{R}$. In [**12**], Titchmarsh proved this was the best possible estimate.

2. ARBITRARILY LARGE POINTWISE GROWTH.

On the real line we have the following example of arbitrarily large pointwise growth.

Proposition. *Given any sequence of positive real numbers $\{a_n\}$, there is a continuous function $f : \mathbb{R} \to \mathbb{C}$ such that $\hat{f}(s)$ exists for each $s \in \mathbb{R}$ and $\hat{f}(n) \geq a_n$ for all $n \geq 1$.*

Proof. Let $\alpha_n = 2a_n + 1$. We can assume that $a_n \geq 1$. Let

$$f_n(x) = \alpha_n e^{-inx} \sin(r_n x) \chi_{[-b_n, b_n]}(x)/x$$

for $x \neq 0$ and $f_n(0) = \alpha_n r_n$, where $0 < r_n \leq 1/2$ are chosen such that $\sum \alpha_n r_n < \infty$. The sequence $b_n > 0$ is to be determined so that $b_n r_n$ is an integer multiple of π. Let $f(x) = (1/\pi) \sum f_n(x)$.

To compute the Fourier transform of f, use the formula

$$\int_0^\infty \sin(ax) \frac{dx}{x} = \begin{cases} \pi/2, & a > 0 \\ 0, & a = 0 \\ -\pi/2, & a < 0. \end{cases} \tag{4}$$

This integral can be evaluated with contour integration [**10**, p. 184], with uniform convergence [**9**, p. 262], and with the Riemann–Lebesgue Lemma [**2**, p. 589].

The estimate $|f_n(x)| \leq \alpha_n r_n$ shows that $\sum \alpha_n f_n(x)$ converges uniformly on $\mathbb{R}$. We can interchange orders of summation and integration in the calculation of $\hat{f}$. Let $m \geq 1$. We then have

$$\begin{aligned} \hat{f}(m) &= \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha_n \int_{-b_n}^{b_n} e^{i(m-n)x} \sin(r_n x) \frac{dx}{x} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha_n \int_0^{b_n} \{\sin[(|m-n| + r_n)x] - \sin[(|m-n| - r_n)x]\} \frac{dx}{x}. \end{aligned}$$

Integration by parts shows that

$$\left| \int_x^\infty e^{iax'} \frac{dx'}{x'} \right| \leq \frac{2}{ax} \quad \text{for all } a, x > 0. \tag{5}$$

Therefore, (4) and (5) ensure that

$$\begin{aligned} \hat{f}(m) &\geq \alpha_m - \frac{4\alpha_m}{\pi r_m b_m} - \frac{2}{\pi} \sum_{n \neq m} \frac{\alpha_n}{b_n} \left[\frac{1}{|m-n| + r_n} + \frac{1}{|m-n| - r_n} \right] \\ &\geq \alpha_m - \frac{4\alpha_m}{\pi r_m b_m} - \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{\alpha_n}{b_n} \\ &\geq a_m \text{ if } r_m b_m \geq 12/\pi \text{ and } b_n \geq 6(2a_n + 1)2^n/\pi \text{ for all } n \geq 1. \end{aligned} \tag{6}$$

If we take $b_n = \lceil 6(2a_n + 1)2^n/\pi \rceil \pi / r_n$ then the conditions in (6) are satisfied and f is continuous on $\mathbb{R}$.

The interchange of summation and integration can be justified as follows: Let $s \in \mathbb{R}$ and $x \geq 1$. Define

$$G(x) = \sum_{n \geq |s|+1} \int_0^x e^{isx'} f_n(x')\, dx'.$$

Since we already know that $\sum \alpha_n f_n(x)$ converges uniformly and that $|\int_0^\infty f_n| = |\int_0^{b_n} f_n| \leq \alpha_n r_n b_n$, our interchange of summation and integration is valid provided that G converges uniformly on $[0, \infty]$ [**9**, Exercise 5 in §5.6]. The Frullani integral formula [**9**, p. 263] says that

$$\int_0^\infty [\cos(px) - \cos(qx)] \frac{dx}{x} = \log\left(\frac{q}{p}\right); \quad p, q > 0. \tag{7}$$

We have $G(x) = [S_1(x) + S_2(x)]/2$ where

$$S_1(x) = \sum_{\substack{b_n < x \\ n \geq |s|+1}} \alpha_n \int_0^{b_n} \left\{\sin\left[(n-s+r_n)x'\right] - \sin\left[(n-s-r_n)x'\right]\right\}$$
$$-i\left\{\cos\left[(n-s+r_n)x'\right] - \cos\left[(n-s-r_n)x'\right]\right\} \frac{dx'}{x'}$$

and

$$S_2(x) = \sum_{\substack{b_n \geq x \\ n \geq |s|+1}} \alpha_n \int_0^{x} \left\{\sin\left[(n-s+r_n)x'\right] - \sin\left[(n-s-r_n)x'\right]\right\}$$
$$-i\left\{\cos\left[(n-s+r_n)x'\right] - \cos\left[(n-s-r_n)x'\right]\right\} \frac{dx'}{x'}.$$

Using (4), (5), (7), and the inequality $0 \leq \log(1+t) \leq t$ $(t \geq 0)$ we obtain

$$S_1(x) = -\sum_{\substack{b_n < x \\ n \geq |s|+1}} \alpha_n \int_{b_n}^{\infty} \left\{\sin\left[(n-s+r_n)x'\right] - \sin\left[(n-s-r_n)x'\right]\right\}$$
$$-i\left\{\cos\left[(n-s+r_n)x'\right] - \cos\left[(n-s-r_n)x'\right]\right\} \frac{dx'}{x'}$$
$$-i\,\alpha_n \log\left[\frac{n-s+r_n}{n-s-r_n}\right]$$

and

$$\begin{aligned} |S_1(x)| &\leq \sum_{n \geq |s|+1} \frac{4\alpha_n}{b_n}\left[\frac{1}{n-s+r_n} + \frac{1}{n-s-r_n}\right] + \frac{2\alpha_n r_n}{n-s-r_n} \\ &\leq 4\sum_{n=1}^{\infty} \alpha_n\,(3/b_n + r_n) \\ &< \infty. \end{aligned}$$

Also,

$$S_2(x) = i \sum_{\substack{b_n \geq x \\ n \geq |s|+1}} \alpha_n \{I(x) + \log[(n-s+r_n)/(n-s-r_n)]\}$$

where

$$I(x) = \int_{x'=x}^{\infty} \left[e^{i(n-s+r_n)x'} - e^{i(n-s-r_n)x'}\right] \frac{dx'}{x'}.$$

Integrating by parts twice gives

$$\begin{aligned} I(x) = &-\frac{1}{ix}\left[\frac{e^{i(n-s+r_n)x}}{n-s+r_n} - \frac{e^{i(n-s-r_n)x}}{n-s-r_n}\right] \\ &-\frac{1}{x^2}\left[\frac{e^{i(n-s+r_n)x}}{(n-s+r_n)^2} - \frac{e^{i(n-s-r_n)x}}{(n-s-r_n)^2}\right] \\ &-2\int_{x'=x}^{\infty}\left[\frac{e^{i(n-s+r_n)x'}}{(n-s+r_n)^2} - \frac{e^{i(n-s-r_n)x'}}{(n-s-r_n)^2}\right]\frac{dx'}{(x')^3}. \end{aligned}$$

The Mean Value Theorem now shows that there are constants c_1, c_2, and c_3, independent of $n \geq |s| + 1$ and $x \geq 1$, such that

$$|I(x)| \leq c_1 r_n + c_2 r_n \int_{x'=x}^{\infty} \frac{dx'}{(x')^2} \leq c_3 r_n.$$

It now follows that $|S_2(x)| \leq \sum \alpha_n(c_3 r_n + 4r_n) < \infty$. Hence, G converges uniformly on $[0, \infty]$. As there is a similar calculation for $x < -1$, our commutation of $\int$ and $\sum$ in the calculation of $\hat{f}$ is valid. This also shows that $\hat{f}(s)$ exists for every $s \in \mathbb{R}$. ∎

The example can be modified so that f is real-valued if we use

$$f_n(x) = \alpha_n \cos(nx) \sin(r_n x) \chi_{[-b_n, b_n]}(x)/x.$$

With essentially the same proof we can have $\hat{f}(\nu_n) \geq a_n$ for any sequence with $\nu_n - \nu_{n+1} \geq \delta$ for all $n \geq 1$ and some $\delta > 0$. And, if instead of the characteristic function $\chi_{[-b_n, b_n]}$ we put in a C^∞ cutoff function and take r_n small enough, then f can be taken to be C^∞ with $\hat{f}(n) \geq a_n$ for all $n \geq 1$. This is very different from the Lebesgue case. When $f \in L^1$, the smoother f is the more rapidly $\hat{f}$ decays; see [**4**, §3.4], and [**14**, p. 45]. By contrast, with conditional convergence, even for smooth f we can have $\hat{f}$ growing at an arbitrarily large rate. Convergence of $\int_{-\infty}^{\infty} f$ is necessary but not sufficient for the existence of $\hat{f}$.

The usual heuristic explanation of the L^1 Riemann–Lebesgue Lemma is that the rapid oscillation of e^{isx} for large $|s|$ makes the positive and negative parts nearly cancel out in the integral for $\hat{f}$. Following the argument in [**5**, p. 98], shows that L^1 functions are well approximated by continuous functions that are themselves nearly constant on small intervals. For small $\epsilon > 0$, there is a continuous function g such that

$$\begin{aligned} \int_{a-\epsilon}^{a+\epsilon} e^{isx} f(x)\,dx &\approx \int_{a-\epsilon}^{a+\epsilon} e^{isx} g(x)\,dx \approx g(a)\int_{a-\epsilon}^{a+\epsilon} e^{isx}\,dx \\ &= 2\,g(a)e^{isa}\sin(s\epsilon)/s \to 0 \quad \text{as } |s| \to \infty. \end{aligned}$$

Summing such terms then shows that $\hat{f} \to 0$ as $|s| \to \infty$. However, with conditionally convergent integrals the integrand can oscillate with nearly the same period as e^{isx} over large intervals. For example, in (1) put $a = 1$. The integrand is then $\cos(x^2)\cos(sx)$. When x is close to s the integrand is close to $\cos^2(x^2)$, which no longer oscillates. Thus, integrating near s contributes a relatively large amount to the integral so that $\hat{f}$ does not go to 0 as $s \to \infty$. Examples in the next section also illustrate this point.

3. MORE FOURIER INTEGRALS. Integrating by parts shows that the integral

$$J = \int_{x=0}^{\infty} x^{\alpha} e^{i[ax^2 - sx]}\, dx \tag{8}$$

converges for $|\alpha| < 1$. Changing the sign of s gives a similar integral. (Letting $\alpha \to 1^-$ then leads to two divergent integrals. In [**3**, 3.851] and [**8**, 2.5.22] they are listed as converging! See [**11**] for a discussion of these divergent integrals.) Our main goal here is to see how J behaves as $s \to \infty$. The integrand can grow nearly linearly. How does this affect the growth of J?

Assume that $0 < \alpha < 1$. Use the transformation $x \mapsto sx/a$. Then

$$J = \left(\frac{s}{a}\right)^{\alpha+1} \int_{x=0}^{\infty} x^{\alpha} e^{it[(x-1/2)^2 - 1/4]}\, dx,$$

where $t = s^2/a$. As $t \to \infty$ the exponential term oscillates rapidly except near the minimum of $x^2 - x$. Thus, we expect nearly perfect cancellation except near $x = 1/2$, and we expect that the major contribution to J should come from integrating near $1/2$. Hence,

$$\begin{aligned} J &\sim \left(\frac{s}{a}\right)^{\alpha+1} 2^{-\alpha} e^{-it/4} \int_{x=1/2-\epsilon}^{1/2+\epsilon} e^{it(x-1/2)^2}\, dx \quad (t \to \infty) \\ &\sim \left(\frac{s}{a}\right)^{\alpha+1} 2^{-\alpha} e^{-it/4} \int_{x=-\infty}^{\infty} e^{itx^2}\, dx \quad (t \to \infty) \\ &= \left(\frac{s}{a}\right)^{\alpha+1} 2^{-\alpha} e^{-it/4} \sqrt{\pi/t}\, e^{i\pi/4} \quad (s^2/a \to \infty) \\ &= \sqrt{\pi}\, 2^{-\alpha} e^{i(\pi - s^2/a)/4} a^{-(\alpha+1/2)}\, s^{\alpha} \quad (s^2/a \to \infty). \end{aligned} \tag{9}$$

Integrating by parts on the complement of $(1/2 - \epsilon, 1/2 + \epsilon)$ and using the Riemann–Lebesgue Lemma shows that (9) gives the dominant behaviour of J. This heuristic argument is made precise in [**13**, pp. 76–84] and is known as the *principle of stationary phase*.

Fixing $a > 0$ and taking α close to 1 shows that the best estimate is $J = O(s^w)$ $(s \to \infty)$, where $w < 1$ but w can be made arbitrarily close to 1. The asymptotic behaviour of J as $\alpha \to 1^-$ and $s \to \infty$ is much more complicated. Such uniform asymptotic approximations are discussed in Chapter VII of [**13**].

The integral J can be evaluated in terms of confluent hypergeometric functions [**6**, pp. 23, 24, 136], parabolic cylinder functions [**7**, pp. 23, 136] and hypergeometric functions [**8**, p. 430]. Changing the sign of s puts the minimum of the exponent $ax^2 + sx$ outside the integration interval. Integration by parts then shows that $\int_0^{\infty} x^{\alpha} e^{i[ax^2+sx]}\, dx = o(s^{\alpha-1})$ $(s \to \infty)$. Linear combinations of this integral and (8) lead to four integrals akin to (1) and (2), whose behaviour as $s \to \infty$ is given by (9).

Let's consider one final integral. Let

$$K = \int_{x=0}^{\infty} x^{\alpha} e^{i[ax^{\nu}-sx]}\,dx$$
$$= \left(\frac{s}{a}\right)^{\frac{\alpha+1}{\nu-1}} \int_{x=0}^{\infty} x^{\alpha} e^{it[x^{\nu}-x]}\,dx \quad \left(t = (s^{\nu}/a)^{1/(\nu-1)}\right). \tag{10}$$

Integration by parts shows that (10) converges for $-1 < \alpha < \nu - 1$. Write $\phi(x) = x^{\nu} - x$. If $\nu > 1$ then ϕ has a minimum at $\nu^{-1/(\nu-1)}$. Expanding near this point gives

$$K \sim \sqrt{\frac{2\pi}{\nu-1}}\, e^{i\pi/4} e^{i(s^{\nu}/a)^{1/(\nu-1)}[\nu^{-\nu/(\nu-1)}-\nu^{-1/(\nu-1)}]} \nu^{-\frac{2\alpha+1}{2(\nu-1)}}\, a^{-\frac{2\alpha+1}{2(\nu-1)}}\, s^{\frac{2\alpha+2-\nu}{2(\nu-1)}}$$

as $t \to \infty$. What values of α and ν make K large when $s \to \infty$? If $\alpha > \nu/2 - 1$, the growth of K is largely determined by how close ν is to 1^+. Fix $\alpha \in (-1/2, 0]$ and fix $a > 0$. It is then apparent that the best estimate of K as $s \to \infty$ is $O(s^w)$, where $w = (\alpha + 1 - \nu/2)/(\nu - 1)$. This exponent can be made arbitrarily large by taking ν close to 1^+. Hence, large growth in K does not come from taking α close to $\nu - 1$ to make the term x^{α} as large as possible as $x \to \infty$; rather, it comes from flattening out the minimum of ϕ by making ϕ nearly linear.

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