Recursive integration of piecewise-continuous functions

D.J. Jeffrey

Department of Applied Mathematics,
The University of Western Ontario,
London, Ontario, Canada N6A 5B7
djj@uwo.ca
http://www.apmaths.uwo.ca/~djj

A.D. Rich

Soft Warehouse Inc.
3660 Waialae Avenue, Suite 304
Honolulu, Hawaii 96816 USA
http://www.derive.com

Abstract

An algorithm is given for the integration of a class of piecewise-continuous functions. The integration is with respect to a real variable, because the functions considered do not in general allow integration in the complex plane to be defined. The class of integrands includes commonly occurring waveforms, such as square waves, triangular waves, and the floor function; it also includes the signum function. The algorithm can be implemented recursively, and it has the property of ensuring that integrals are continuous on domains of maximum extent.

1 Introduction

The integration of a function expressed using the Maple function `piecewise` or the signum function was considered in [3], where the fundamental definitions and theorems on integrating discontinuous functions were presented. We recall that a function $F(x)$ is said to have breakpoints at those values of x where the function is discontinuous. It was also pointed out in [3] that the problem of integrating piecewise-continuous functions can be posed only within the context of integration with respect to a real variable, and thus we continue to work in that context. The integration problem has two aspects: deriving a primitive, or anti-derivative for a given function, and ensuring that the result returned by the integrator is valid on a domain of maximum extent [2].

We also follow [3] in noting that a discussion of the integration of piecewise-continuous functions can be distracted by contentious, but irrelevant, issues, such as which definitions of the signum and Heaviside functions are the correct ones. These issues cannot be ignored completely, because the value of sgn(0) has a bearing on the results given here. However, we shall avoid the distraction by defining a cognate of the signum function that fulfills the requirements of integration, and use it without prejudice to the wider discussion.

The new features of the present algorithm are, first, an extension to a broader class of integrands. Specifically, it can handle piecewise-continuous functions with periodic breakpoints, for example, the floor function and square waves. Second, the manner in which the computation is performed is modified. The previous algorithm [3] put the finite number of breakpoints into an ordered list. The new algorithm works recursively and avoids this step, which is in any event no longer possible in the presence of periodic breakpoints.

2 Class of integrands

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function. Denote this function by $F(x, \{p_i | i = 1 \ldots n\})$, where x is regarded as the independent variable, and the $p_1 \ldots p_n$ are regarded as constants. A piecewise-continuous function is now constructed by replacing the constants p_i with piecewise-continuous functions P_i selected from the list below. Each of the P_i can have a linear real argument in x. The functions resulting from this construction constitute the class of integrands addressed.

The cognate of the signum function that is most convenient for expressing the results of integration is defined by

$$S(x) = \begin{cases} 1\,, & \text{for } x \geq 0, \\ -1\,, & \text{for } x < 0. \end{cases} \tag{1}$$

For brevity in this paper, S will still be called a signum function. The Heaviside function $H(x) : \mathbb{R} \to \mathbb{Z}$ is defined by

$$H(x) = \begin{cases} 1\,, & \text{for } x \geq 0, \\ 0\,, & \text{for } x < 0. \end{cases} \tag{2}$$

The floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$ is the largest integer less than or equal to x. The ceiling function $\lceil x \rceil = -\lfloor -x \rfloor$ is the smallest integer greater than or equal to x. The rounding function $\mathcal{R}(x)$ is defined by $\mathcal{R}(x) = \lfloor x + \frac{1}{2} \rfloor$.

Three waveforms commonly studied in engineering are the square wave $SQW(x)$, the rising sawtooth wave $STW(x)$ and the triangular wave $TRW(x)$, defined by

$$SQW(x) = S(\sin \pi x)\,, \tag{3}$$

$$STW(x) = (-2/\pi) \arctan(\cot \pi x)\,, \tag{4}$$

$$TRW(x) = SQW(x) STW(x)\,. \tag{5}$$

These three functions have their breakpoints at integers and have the range $[-1, 1]$. The real-valued versions of the absolute-value function and the modulus function are also included in the list.

3 Preliminary transformations

This section establishes that an integrand containing any of the above functions can be reduced to one containing only floor and signum. These transformations are not in general equalities, because of pointwise differences, but they leave any integral unchanged.

$$\begin{aligned} |x| &\Rightarrow xS(x) \ . \\ x \bmod y &\Rightarrow x - y\lfloor x/y \rfloor \ . \\ H(x) &\Rightarrow \tfrac{1}{2}(1+S(x)) \ . \\ SQW(x) &\Rightarrow \cos(\pi\lfloor x \rfloor) \ . \\ STW(x) &\Rightarrow 2x - 1 - 2\lfloor x \rfloor \ . \end{aligned}$$

In addition, if a signum function has a nonlinear argument that can be factored, then we have the transformation

$$S((ax-b)g(x)) \Rightarrow S(ax-b)S(g(x)) \ .$$

Finally, the argument of S can be simplified by using

$$S(ax-b) \Rightarrow S(a)S(x-b/a) \ ,$$

where it is assumed that $a \neq 0$.

Therefore, any integration problem $\int f(x,\{P_i\})dx$ can be replaced by the problem

$$\int F(x, \{S(x-b_i)|i=1..p\}, \{\lfloor c_j x - d_j \rfloor | j = 1..q\})dx \ , \quad (6)$$

where for nontriviality it is assumed that $c_j \neq 0$.

4 Integration procedure

The integration problem (6) is solved in a way that can be programmed recursively. Integration starts by replacing all of the signum and floor functions by temporary symbolic constants; here symbols s_i are used for S functions and f_j for floor:

$$F \Rightarrow F(x, \{s_i|i=1..p\}, \{f_j|j=1..q\})\ dx \ .$$

The standard procedures of the particular system can now be used to integrate $F(x,\{s_i\},\{f_j\})$ with respect to x to obtain the primitive $G(x,\{s_i\},\{f_j\})$.

Theorem: Let $G(x,\{s_i\},\{f_j\})$ be a primitive of the function $F(x,\{s_i\},\{f_j\})$, and let G be continuous in all its arguments. Then $G(x,\{S(x-b_i)\},\{\lfloor c_j x - d_j\rfloor\})$ is a primitive of $F(x,\{S(x-b_i)\},\{\lfloor c_j x - d_j \rfloor\})$.

Proof: The set of breakpoints of F is

$$\mathbb{D} = \{b_i|i=1..p\} \cup \{(n+d_j)/c_j | j=1..q, n \in \mathbb{Z}\}.$$

For any $x \notin \mathbb{D}$, there exists a neighbourhood of x within which the signum and floor functions are constant. Hence $G' = F$ within that neighbourhood by construction. Thus $G' = F$ for all $x \notin \mathbb{D}$. □

Although the function $G(x,\{s_i\},\{f_j\})$ is assumed to be continuous, after the temporary constants are returned to their signum and floor functions, there will be discontinuities at the breakpoints in general. To obtain an integral valid on the domain of maximum extent [2], these discontinuities must be removed. The function G can be regarded as a candidate for the integral of F, but one that must be rectified, i.e. made continuous. The algorithm continues by choosing one of the temporary constants and substituting the original function for it, and then eliminating any discontinuities thereby introduced before continuing to the next constant.

Starting with the first signum, we proceed as follows.

$$G(x,\{s_i\},\{f_j\}) \Rightarrow G(x, S(x-b_1), \{s_i|i=2..p\}, \{f_j\}).$$

Now compute

$$\begin{aligned} J &= G(b_1, 1, \{s_i|i=2..p\}, \{f_j\}) \\ &\quad - G(b_1, -1, \{s_i|i=2..p\}, \{f_j\}) \ , \end{aligned}$$

and define

$$\begin{aligned} &G_1(x, \{s_i|i=2..p\}, \{f_j\}) = \\ &\quad G(x, S(x-b_1), \{s_i|i=2..p\}, \{f_j\}) - \tfrac{1}{2} J S(x-b_1) \ . \end{aligned}$$

Theorem: The function G_1 is a primitive of F and is continuous at $x = b_1$.

Proof: If $x \neq b_1$, then $S(x-b_1)$ is a constant and therefore $G_1' = G'$. By direct computation

$$\begin{aligned} \lim_{x\to b_1+} G_1 = \lim_{x \to b_1 -} G_1 &= \tfrac{1}{2} G(b_1, 1, \{s_i|i=2..p\}, \{f_j\}) \\ &+ \tfrac{1}{2} G(b_1, -1, \{s_i|i=2..p\}, \{f_j\}) \ , \end{aligned}$$

where the continuity of G has been used. □

The function G is now discarded and G_1 used instead. Thus G_1 is the new candidate for the integral. As each s_i is returned to $S(x-b_i)$, a new function G_i is created and used in subsequent steps. After p steps, all the constants $\{s_i\}$ have been returned to their signum functions and a function G_p has been computed that is continuous at all breakpoints, and is the candidate for an integral.

A more elaborate procedure is needed in order to return the floor functions to the candidate function. Again the first constant is replaced:

$$G_p(x,\{f_j\}) \Rightarrow G_p(x, \lfloor c_1 x - d_1 \rfloor, \{f_j|j=2..q\}) \ .$$

The floor function is discontinuous when its argument equals an integer, when $c_1 x - d_1 = n$. Therefore calculate the jump

$$\begin{aligned} J_n &= G_p\left(\frac{n+d_1}{c_1}, n, \{f_j|j=2..q\}\right) \\ &\quad - G_p\left(\frac{n+d_1}{c_1}, n-1, \{f_j|j=2..q\}\right) \ , \end{aligned}$$

and define a new candidate function by

$$\begin{aligned} G_{p+1}(x, \{f_j|j=2..q\}) &= G_p(x, \lfloor c_1 x - d_1 \rfloor, \{f_j|j=2..q\}) \\ &\quad - \sum_{m=1}^{\lfloor c_1 x - d_1 \rfloor} J_m \ . \end{aligned} \quad (7)$$

If $\lfloor c_1 x - d_1 \rfloor < 0$, the summation in this formula is evaluated using the convention of Graham, Knuth and Patashnik [1], given in the answer to their exercise 2.1, to wit, if $k < j$, then

$$\sum_{m=j}^{k} P(m) = - \sum_{m=k+1}^{j-1} P(m) \ . \quad (8)$$

Notice that for $k = j - 1$, corresponding in the definition of G_{p+1} to $\lfloor c_1 x - d_1 \rfloor = 0$, the sum must be 0 to be consistent with (8). Obviously, if possible, the summation will be evaluated in finite terms [5].

Theorem: The function G_{p+1} is a primitive of F and is continuous at all the breakpoints of $\lfloor c_1 x - d_1 \rfloor$.

Proof: As with the previous theorem, for $x \notin \mathbb{D}$, the summation is constant and makes no contribution to the derivative of G_{p+1}. Direct computations again show that for each n the two-sided limit at $x = (n + d_1)/c_1$ exists. □

After the procedure is repeated for all constants, a function $G_{p+q}(x)$ has been computed that is a primitive of F and is continuous at all breakpoints of the piecewise-continuous functions.

5 Some notes on implementation

It should be noted that the above procedure does not require that the different signum and floor functions have distinct breakpoints. The procedure continues to apply if some of the breakpoints of different floors or signums coincide. Since each discontinuity is removed with computations based on the latest candidate, the influence of a breakpoint does not last more than a step.

The description in the previous section is not in the form most natural for implementing in a computer algebra system. A recursive way to program the method is to identify one piecewise-continuous function, then either replace it with an equivalent function, following section 2, or generate a temporary name, set domain information and call the integration routine recursively on the new function. When the integration returns from the call, it will give a candidate for an integral. The function that is received is corrected for the discontinuities at the breakpoints of the particular piecewise-continuous function that was replaced. The newly computed candidate integral is then the return value for the integrate routine. If there are more functions that were replaced earlier in the recursive stack, the function will be altered further.

If the summation cannot be computed in closed terms, the integration routine should return the sum anyway. The integral will then correctly plot and evaluate numerically, even with summations.

6 A test suite

We begin by integrating the floor function itself.

$$\int \lfloor x \rfloor \, dx \Rightarrow \int f \, dx = fx \ .$$

The breakpoints are at $x = n$ and therefore the jump function is $J_n = n(n) - n(n-1) = n$. The integral becomes

$$x\lfloor x \rfloor - \sum_{m=1}^{\lfloor x \rfloor} m = x\lfloor x \rfloor - \tfrac{1}{2}\lfloor x \rfloor(\lfloor x \rfloor + 1) \ ,$$

and therefore the result is

$$\int \lfloor x \rfloor \, dx = (x - \tfrac{1}{2})\lfloor x \rfloor - \tfrac{1}{2}\lfloor x \rfloor^2 \ . \tag{9}$$

A variation is

$$\int 12x\lfloor x \rfloor \, dx \Rightarrow \int 12x f_1 \, dx = 6x^2 f_1 \ .$$

The result is

$$\int 12x\lfloor x \rfloor \, dx = (6x^2 - 1)\lfloor x \rfloor - 3\lfloor x \rfloor^2 - 2\lfloor x \rfloor^3 \ . \tag{10}$$

The next example will be done two different ways to illustrate different points. From first principles, it is easy to see that

$$\int SQW(x) \, dx = \tfrac{1}{2} TRW(x) \ . \tag{11}$$

Applying the procedure given here, we obtain

$$\int SQW(x) dx \Rightarrow \int \cos \pi \lfloor x \rfloor dx \Rightarrow \int \cos \pi f dx = x \cos \pi f \ .$$

Therefore $J_n = n(-1)^n - n(-1)^{n+1} = 2n(-1)^n$. The integral becomes

$$\begin{aligned} & x\cos(\pi\lfloor x \rfloor) - \sum_{m=1}^{\lfloor x \rfloor} 2m(-1)^m \\ & = \tfrac{1}{2}\cos(\pi\lfloor x \rfloor)(2x - 1 - 2\lfloor x \rfloor) + \tfrac{1}{2} = \tfrac{1}{2} TRW(x) + \tfrac{1}{2} \ . \end{aligned}$$

The constant $\frac{1}{2}$ can be dropped. A CAS would leave the result in terms of floor.

Integral (11) also provides a test of an integrand containing break points that coincide. This is because another representation of $SQW(x)$ is

$$SQW(x) = 1 + 4\lfloor \tfrac{1}{2}x \rfloor - 2\lfloor x \rfloor \ .$$

Therefore the integral becomes

$$\int (1 + 4f_1 - 2f_2) dx = x + 4f_1 x - 2f_2 x \ .$$

The breakpoints for $f_1 = \lfloor x/2 \rfloor$ are $x = 2n$, so the first fixup becomes $J_n = 8n$ and we obtain

$$x + 4\lfloor \tfrac{1}{2}x \rfloor x - 2f_2 x - 4\lfloor \tfrac{1}{2}x \rfloor(\lfloor \tfrac{1}{2}x \rfloor + 1).$$

After including the fixup for f_2, we obtain

$$x + 4(x-1)\left\lfloor \frac{x}{2} \right\rfloor - 4\left\lfloor \frac{x}{2} \right\rfloor^2 + \lfloor x \rfloor^2 + (1 - 2x)\lfloor x \rfloor.$$

This differs from (11) by the factor $\frac{1}{2}$.

The next example checks the coincidence of breakpoints between signum and floor functions. In this example, they coincide at 0.

$$\int 2S(x)\lfloor x \rfloor \, dx \Rightarrow \int 2s_1 f_1 dx = 2s_1 f_1 x \ .$$

If the signum fix-up is done first, it is 0. So we have $G = 2S(x)f_1 x$. The second fix-up gives

$$2S(x)\lfloor x \rfloor x - S(x)\lfloor x \rfloor(\lfloor x \rfloor + 1) \ .$$

The same result is obtained by performing the fix-ups in the opposite order.

In electrical engineering, the half-wave rectified sinewave is studied [7]. This is described by the function

$$HW(x) = (SQW(x) + 1)\sin \pi x \ .$$

Using the methods given here, its integral is given by

$$\int HW(x)\,dx = \frac{1}{\pi}(2\lfloor x\rfloor - \cos\pi x(1+\cos\pi\lfloor x\rfloor)) \ . \qquad (12)$$

Similarly the full-wave rectified sinewave is studied. It is defined to be $FW(x) = SQW(x)\sin\pi x$. However it can be equivalently expressed as $\sqrt{1-\cos 2\pi x}/\sqrt{2}$, in which form it was studied in [4], where an integral was obtained that is equivalent to that obtained by the present method.

$$\int FW(x)\,dx = \frac{2\lfloor x\rfloor}{\pi} - \frac{\cos\pi x\cos\pi\lfloor x\rfloor}{\pi} \ . \qquad (13)$$

In [4], continuous integrals were obtained for trigonometric functions by including floor functions in the final results. Could there be a difficulty if an integrand studied in [4] additionally had a floor function added to it? To be specific, the following integral was evaluated using the Weierstrass $\tan(x/2)$ substitution and the rectification described in [4].

$$\begin{aligned}\int\sqrt{1-\cos x}\,dx &= S(\sin x)\left(2\sqrt{2}-2\sqrt{1+\cos x}\right)\\ &\quad +4\sqrt{2}\left\lfloor\frac{x+\pi}{2\pi}\right\rfloor \ .\end{aligned}$$

We now modify this problem by multiplying by a square wave. After applying the procedures of this paper, the final answer is correctly obtained as

$$\begin{aligned}&\int SQW(x/2\pi)\sqrt{1-\cos x}\,dx =\\ &\quad (-1)^{\lfloor x/2\pi\rfloor}\left[S(\sin x)\left(2\sqrt{2}-2\sqrt{1+\cos x}\right)\right.\\ &\quad \left.+4\sqrt{2}\left\lfloor\frac{x+\pi}{2\pi}\right\rfloor - 4\sqrt{2}\left\lfloor\frac{x}{2\pi}\right\rfloor - 2\sqrt{2}\right] \ . \qquad (14)\end{aligned}$$

This expression could be simplified further, but the algebraic simplification of these functions is not a topic of this paper.

The summation required to rectify integrals containing floor may require functions not supported by the system. For example, if the Ψ, or digamma, function is not available then

$$\int\frac{\lfloor x\rfloor}{x^2}\,dx = -\frac{\lfloor x\rfloor}{x} + \sum_{m=1}^{\lfloor x\rfloor}\frac{1}{m} \ . \qquad (15)$$

If Ψ is supported then the integral is $-\lfloor x\rfloor/x + \Psi(1+\lfloor x\rfloor)$. Notice that both expressions are valid only for $x>0$. Since the integral does not exist at the origin, the summation term cannot be connected to $x<0$. Therefore, a user who wishes to evaluate definite integrals on the negative real axis, or plot the integral there, needs a mechanism for telling the system of the intended domain of x. This is another very general topic in computer algebra. Clearly in this case we need

$$\int\frac{\lfloor x\rfloor}{x^2}\,dx = -\frac{\lfloor x\rfloor}{x} + \sum_{m=-1}^{\lfloor x\rfloor}\frac{1}{m} \text{ with proviso } x<0 \ .$$

This can be summed using (8) to obtain

$$\int\frac{\lfloor x\rfloor}{x^2}\,dx = \frac{\lfloor x\rfloor}{x^2} + \Psi(-\lfloor x\rfloor) \ .$$

This last example points to a small refinement of the integration procedure regarding the starting value of the summation in (7). Given a definite integral on the domain $[\alpha,\beta]$, the summation at step $p+i$ should start at $m=\lfloor c_i\alpha+d_i\rfloor$.

7 Nesting and discontinuities

Although section 4 did not allow nested signums and floors, there are two reasons for considering them. First, the example $\big|2-|x|\big|$ was integrated in [3]. Second, a straightforward recursive implementation might accept them. This comes about as follows. Rewrite the above problem as

$$\big|2-|x|\big| = S(2-S(x)x)(2-S(x)x) \ .$$

An implementation would reject the outer signum as not having a linear argument, but it would accept the inner one and replace it. Therefore the integrate routine would be called on $S(2-s_1x)(2-s_1x)$. Next time through the integrate routine, the argument of S appears to be linear, unless the system can distinguish s_1 as being different from other symbols. Therefore, the system integrates $s_2(2-s_1x)$ to obtain

$$G = s_2(2x - \tfrac{1}{2}s_1x^2) \ .$$

The breakpoint for $s_2 = S(2-s_1x)$ is $x=2/s_1$. The fixup for s_2 is $J=4/s_1$, giving

$$G = S(2-s_1x)(2x-\tfrac{1}{2}s_1x^2) - (2/s_1)S(2-s_1x) \ .$$

The breakpoint is now $x=0$ where $J=-4$. Therefore the integral is

$$\begin{aligned}\int\big|2-|x|\big|\,dx &= S(2-S(x)x)(2x-\tfrac{1}{2}x^2S(x))\\ &\quad -2S(x)S(2-S(x)x)+2S(x) \ . \qquad (16)\end{aligned}$$

This differs by the constant 4 from the integral in [3].

Another example showing the composition of functions is

$$\begin{aligned}&\int 4\lfloor xS(2x+3)\rfloor\,dx = 4x\lfloor xS(2x+3)\rfloor\\ &-S(2x+3)(2\lfloor xS(2x+3)\rfloor^2 + 2\lfloor xS(2x+3)\rfloor + 5) \ . \qquad (17)\end{aligned}$$

In many cases, however, nesting will lead to functions that violate the continuity requirements of section 4. For example, $S(2-\lfloor x\rfloor x)$, $x^{\lfloor x\rfloor}$ and $x^{S(x)}$. A thorough implementation, therefore, must either restrict itself to a safe class of integrands, such as those in which signum and floor appear only as multiplicative or additive terms, or test the result of integration for continuity. The theorems in section 4 give sufficient, not necessary, conditions for the procedure to succeed. Thus it is possible to integrate $x^{S(x)+1/2}$ even though $x^{s_1+3/2}/(s_1+3/2)$ is not continuous for all s_1.

8 Closing remarks

Some integrals, for example the integral of $x^{S(x)}$, are better handled using the methods of the previous paper [3], while the examples given in section 6 show the advantages of the present method. The method is a rectifying transform in the sense of [2], and requires a knowledge of the points of discontinuity of the integral. In the present case, these coincide with the breakpoints of the component piecewise-continuous functions, which suggests that the restriction of the functions to linear arguments is not essential. However, from a practical point of view, it is unlikely that more elaborate arguments will arise often, and so they have not been pursued here.

This paper has focussed on the integration of piecewise-continuous functions. We do not wish to imply by this that the algebra of these functions could not benefit from further work. Rather the opposite: the utility of these functions, something long recognised by engineers and physicists, warrents more attention on the part of CAS developers. While working with integrals of piecewise-continuous functions, a number of algebraic identities were generated that show how difficult algebraic simplification can become. Even before integration, the alternative representations of $SQW(x)$, namely

$$SQW(x) = \cos(\pi\lfloor x\rfloor) = (-1)^{\lfloor x\rfloor} = 1 + 4\lfloor x/2\rfloor - 2\lfloor x\rfloor \ ,$$

could be used to present a system with difficult problems in identifying zero. Integrating this equation led to identities between the triangular wave and floor functions which again would challenge a CAS. The uniqueness of normal forms has been discussed in [6].

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