A survey of nonabsolute integration

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Outline:

Why we need nonabsolute integrals
Some examples
The Henstock integral on the real line
The Fundamental Theorem of Calculus
Multipliers
Examples and generalizations

Why we need nonabsolute integrals

The following theorem is false for Riemann and Lebesgue integrals:

Fundamental Theorem of Calculus (?)
Let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then for all x in $[a, b]$

$$\int_{y=a}^{x} F'(y)\, dy = F(x) - F(a).$$

EXAMPLE

$$F(x) = \begin{cases} x^2 \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$F'(x) = \begin{cases} 2x \sin(x^{-2}) - 2x^{-1} \cos(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\int_0^1 F'(y)\, dy$ does not exist as a Riemann or Lebesgue integral

Also false for Riemann and Lebesgue integrals:

Stokes's Theorem(?)

Let $\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be differentiable. Let S be a smooth surface and ∂S be its boundary. Then

$$\int_{\partial S} \vec{A} \cdot d\vec{r} = \int_S \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, dS$$

Nonabsolute integrals

$$\int_0^\infty y^N \sin(e^y)\, dy = \int_1^\infty (\log x)^N \sin(x) \frac{dx}{x}$$

Dirichlet's Theorem:

If there is $M \in \mathbb{R}$ such that $|\int_1^A f(x)\, dx| \leq M$ for each $A > 1$ and g is of bounded variation with $\lim_{x\to\infty} g(x) = 0$ then $\int_1^\infty f(x)g(x)\, dx$ exists.

Bounded variation: There exists $V \in \mathbb{R}$ such that for any set of disjoint intervals $\{(x_{i-1}, x_i)\}$ we have $\sum |g(x_i) - g(x_{i-1})| < V$.

Henstock integral (Ralph Henstock 1961, Jaroslav Kurzweil 1957)

$f\colon[-\infty,\infty]\to\mathbb{R}$ convention $f(\pm\infty)=0$

Partition of $[-\infty,+\infty]$
$-\infty=x_0<x_1<\cdots<x_N=+\infty$

Tagged partition
$\mathcal{P}=\{(z_i,I_i)\}_{i=1}^N$ where
$I_i=[x_{i-1},x_i]$ and $z_i\in I_i$

Gauge
$\gamma\colon[-\infty,+\infty]\to\{$open intervals in $[-\infty,+\infty]\}$
$\gamma(x)$ is an open interval containing x

$\mathcal{P}$ is γ-fine if $\gamma(z_i)\supset I_i$ for each $1\le i\le N$

Open intervals:
$[-\infty,a),(a,b),(b,+\infty],[-\infty,+\infty]$
for all $-\infty\le a<b\le+\infty$

f is Henstock integrable, $\int\limits_{-\infty}^{+\infty} f = L \in \mathbb{R}$, if and only if

$(\forall \epsilon > 0)(\exists \gamma)$ if $\mathcal{P}$ is any γ-fine tagged partition of $[-\infty, +\infty]$ then

$$\left| \sum_{i=1}^{N} f(z_i)|I_i| - L \right| < \epsilon,$$

where $|I_i|$ is the length of I_i.

Convention:

$0 \cdot \infty = 0$

Basic properties:

If $f \in L^1$ then f is Henstock integrable

If f is improper Riemann integrable then f is Henstock integrable

There are no improper integrals:

$$\int_{-\infty}^{+\infty} f = L \iff \int_{-\infty}^{A} f \text{ exists for each } -\infty < A < +\infty \text{ and } \lim_{A \to +\infty} \int_{-\infty}^{A} f = L$$

Fundamental Theorem of Calculus

(i) Let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) except on a countable set. Then F' is Henstock integrable and $\int\limits_{y=a}^{x} F'(y)\, dy = F(x) - F(a)$ for all $x \in [a, b]$.

(ii) Let $f : [a, b] \to \mathbb{R}$ be Henstock integrable. Define $F(x) = \int_a^x f$. Then $F'(x) = f(x)$ for almost all $x \in (a, b)$.

Henstock integrals are nonabsolute

$$\int_a^b f \text{ exists} \quad \not\Longrightarrow \quad \int_a^b |f| \text{ exists}$$
$$\int_a^b |f| \text{ exists} \quad \not\Longrightarrow \quad \int_a^b f \text{ exists}$$

EXAMPLE

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

f is not Riemann integrable over $[0,1]$ but $|f|$ is!

EXAMPLE

Let $A \subset [0,1]$ be a nonmeasurable set

$$f(x) = \begin{cases} 1, & x \in A \\ -1, & x \notin A \end{cases}$$

f is not Lebesgue integrable over $[0,1]$ but $|f|$ is!

Multipliers

Banach-Steinhaus:
Suppose E is a measurable set and the functions g_α are bounded. If for each $f \in L^1$ we have $|\int_E f g_\alpha| < M(f)$ then $|g_\alpha| < C$.

Henstock integrals:
Let g be measurable. If $\int_I fg$ exists for all Henstock integrable functions f then g is of bounded variation.

Higher dimensions

Integrate over a rectangle in $\mathbb{R}^2$
-Riemann-squares
-refinement
-rectangles

Figures:
-finite unions of rectangles that satisfy a regularity condition:

$$\frac{\text{area}}{\text{diameter} * \text{perimeter}} > \lambda$$

Distributions and more general integrands

Measure space $(X, \mathcal{M}, \mu)$
$f : E \to \mathbb{R} \qquad (E \in \mathcal{M})$

$$\int_E f \, d\mu \sim \sum_{i=1}^{N} f(z_i)\mu(I_i)$$

$f(z)\mu(I)$ takes pair (point in X, set in $\mathcal{M}$) $\to \mathbb{R}$

More generally,
$h : \mathbb{R} \times \{$algebra of intervals in $\mathbb{R}\} \to \mathbb{R}$

$$\int_I h \sim \sum_{i=1}^{N} h(z_i, I_i)$$

EXAMPLE
$\delta =$ Dirac measure

$$\delta(E) = \begin{cases} 1, & 0 \in E \\ 0, & 0 \notin E \end{cases}$$

$$\int_E f \, d\delta = \begin{cases} f(0), & 0 \in E \\ 0, & 0 \notin E \end{cases}$$

EXAMPLE

$\delta' =$ derivative of Dirac measure

$\phi\colon \mathbb{R} \to \mathbb{R}$ is a test function
(C^∞, compact support)

$$\delta'[\phi] = -\phi(0)$$

$$h_f(z, E) = \begin{cases} -f'(0), & 0 \in E \\ 0, & 0 \notin E \end{cases}$$

$$\int_E f\, d\delta' = \int_E h_f = \begin{cases} -f'(0), & 0 \in E \\ 0, & 0 \notin E \end{cases}$$

Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Theorem (Denjoy)

If $b_n \downarrow 0$ but $\sum \frac{b_n}{n} = \infty$ then f is not L^1 and not Henstock integrable.

EXAMPLE

$$\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n}$$

Symmetric integrals

Partition: $a < x_1 < x_2 < \cdots < x_N < b$

Endpoint symmetry: $x_1 - a = b - x_N$

γ-fine: $\gamma(\frac{1}{2}(x_i + x_{i+1})) \supset [x_i, x_{i+1}]$
$\gamma(a) \supset [a, x_1]$
$\gamma(b) \supset [x_N, b]$

$$\int_a^b f \sim \sum_{i=1}^{N} f(\tfrac{1}{2}(x_i + x_{i+1}))[x_{i+1} - x_i]$$

If $f(x) = \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$
then

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \end{aligned}$$

Alexiewicz norm:

$$\|f\| = \sup_{I \subseteq [a,b]} \left| \int_I f \right|$$

-not complete
-barrelled space

Harmonic functions on the disc

Dirichlet problem:

$$\Delta u = 0 \quad \text{for} \quad |z| < 1$$
$$u(e^{i\theta}) = f(\theta) \quad \text{for} \quad |\theta| \le \pi$$

Poisson integral

$$u(re^{i\theta}) = \frac{1-r^2}{2\pi} \int_{\phi=-\pi}^{\pi} \frac{f(\phi)\,d\phi}{1 - 2r\cos(\phi - \theta) + r^2}$$

Let $u_r(\theta) = u(re^{i\theta})$. Then

$\|u_r\| \le \|f\|$ for all $0 \le r < 1$

$\|u_r - f\| \to 0$ as $r \to 1^-$

$\|u_r\|_1 = o\left(\frac{1}{1-r}\right)$ as $r \to 1^-$

Newtonian potential

$$V(x) = \frac{1}{4\pi} \int\limits_{x' \in \mathbb{R}^3} \frac{f(x')\,dx'}{|x - x'|} \qquad \begin{array}{l} x = (x_1, x_2, x_3) \\ x' = (x'_1, x'_2, x'_3) \end{array}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$

Poisson equation

$$\Delta u = -f$$

f is the charge density
$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

Electric field
$\vec{E} = -\vec{\nabla} V$
Force on a charged particle with charge q is
$\vec{F} = q\vec{E}$

$$u(x) = V(x) = \frac{1}{4\pi} \int\limits_{x' \in \mathbb{R}^3} \frac{f(x')\,dx'}{|x - x'|}$$

V exists on $\mathbb{R}^3$ if and only if

$$\int\limits_{B(x,1)} \frac{f(x')\,dx'}{|x - x'|} < \infty \quad \text{for each} \quad x \qquad (1)$$

$$\int\limits_{\mathbb{R}^3} \frac{f(x')\,dx'}{1 + |x'|} < \infty \qquad (2)$$

Compact support

$$f = 0 \text{ for } |x| > R \Rightarrow V(x) = O\left(\frac{1}{|x|}\right) \quad (|x| \to \infty)$$

Question What are necessary and sufficient conditions on f so that (1) and (2) hold and $V(x) = O(|x|^{-1})$?

Necessary and sufficient condition

$V(x) = O(|x|^{-1})$ if and only if

$$\int_{\rho=0}^{\rho_0} \int_{B(x,\rho|x|)} f(x')\, dx' \, \frac{d\rho}{\rho^2} < M$$

for some $\rho_0 > 1$