

A certain series associated with Catalan's constant

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Abstract

A parametric class of series generated by integration of complete elliptic integrals

$$\sum_{\substack{k=0 \\ k \neq -r}}^{\infty} \frac{\binom{2k}{k}^2}{(k+r) 16^k}$$

is evaluated in closed form. Alternative proofs to results of Ramanujan and others are given. A particular case of the Saalschutzhian hypergeometric series ${}_4F_3(1)$ is derived.

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0. Preamble

The subject of our interest is the hypergeometric series generated by elliptic integrals

$$S(r) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)16^k} = \frac{1}{r} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, r; 1, r+1; 1\right) \quad (1)$$

This series has a long and interesting story. About a century ago Ramanujan [1, p. 351 and 2, p. 39] in his first letter to Hardy stated without proof a particular case of (1), when the parameter $r = n$ is a positive integer, namely

$$S(n) = \frac{16^n}{\pi n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k} \quad (2)$$

In 1927, when Ramanujan's collected papers were published and result (2) became publicly known, it attracted a great deal of attention. Different proofs were given by Watson [3] and Darling [4], later Bailey [5] and Hodgkinson [6] generalized (2) to

$${}_3F_2(a, b, c+n-1; c, a+b+n; 1) = \frac{\Gamma(n)\Gamma(a+b+n)}{\Gamma(a+n)\Gamma(b+n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k}{k! (c)_k} \quad (3)$$

which gives Ramanujan's result when $a = b = \frac{1}{2}$ and $c = 1$. Ramanujan [7, pp. 237-239 and 2, p. 45] also stated a complementary formula to (2), when the parameter $r = n + \frac{1}{2}$ is a half integer, namely

$$S\left(n + \frac{1}{2}\right) = \frac{4}{\pi} \frac{\binom{2n}{n}^2}{16^n} \left(2G + \sum_{k=0}^{n-1} \frac{16^k}{\binom{2k}{k}^2 (2k+1)^2} \right). \quad (4)$$

Here G is Catalan's constant defined by

$$G = \frac{1}{2} \int_0^1 \mathbf{K}(k) dk = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

and \mathbf{K} is the complete elliptic integral of the first kind, given by

$$\mathbf{K}(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

As mentioned in [2, p. 47], Ramanujan's proofs of formulas (2) and (4) most likely were based on the recurrence equation

$$\left(r + \frac{1}{2}\right)^2 S(r+1) - r^2 S(r) = \frac{1}{\pi} \quad (5)$$

subject to initial conditions. This equation is derived from the fact that $S(r)$ is generated by integration of complete elliptic integrals as

$$S(r) = \frac{2}{\pi} \int_0^1 z^{r-1} \mathbf{K}(z) dz, \quad \operatorname{Re}(r) > 0. \quad (6)$$

In 1981, unaware of Ramanujan's equation (5), Dutka [8] employed by (6) rediscovered formulas (2) and (4). In section 3 we outline the derivation of equation (5), as well as its solution. In view of (5), it's pretty straightforward to see that for any rational $r = n + p$, where n is a positive integer and $0 < p \leq 1$, series (1) has a closed form representation

$$S(n+p) = \frac{(p)_n^2}{(p + \frac{1}{2})_n^2} \left(S(p) + \frac{1}{\pi p^2} \sum_{k=0}^{n-1} \frac{(p + \frac{1}{2})_k^2}{(p+1)_k^2} \right) \quad (7)$$

Here $(p)_n = p(p+1) \dots (p+n-1)$ is the Pochhammer symbol. There are only three known cases when the function $S(p)$ is expressible in terms other than hypergeometric functions, namely $p = 1, \frac{1}{2}, \frac{1}{4}$.

$$S(1) = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; 1, 2; 1\right) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; 1\right) = \frac{4}{\pi}$$

$$S\left(\frac{1}{2}\right) = 2 {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; 1\right) = \frac{8G}{\pi}$$

$$S\left(\frac{1}{4}\right) = 4 {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}; 1, \frac{5}{4}; 1\right) = \frac{\Gamma(\frac{1}{4})^4}{4\pi^2}$$

where $\Gamma(z)$ is the Euler gamma function. All these cases are due to Ramanujan. L.Glasser [9] made a conjecture that it is possible to express $S(\frac{1}{2^k})$ for $k \geq 3$ in finite terms, however that is remained to be seen.

It does not appear to have been previously studied the case when the parameter r in (1) is a negative integer (assuming that the term $r = -k$ is dropped from summation):

$$S(r) = \sum_{\substack{k=0 \\ k \neq -r}}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)16^k}. \quad (8)$$

A few particular cases of (8) appeared in the handbooks by E.P. Adams and R.L. Hippisley [10] and by E.R. Hansen [11]:

$$S(-1) = -\frac{2G+1}{\pi} + \log(2) - \frac{1}{2}$$

$$S(-2) = -\frac{18G+13}{16\pi} + \frac{9}{16} \log(2) - \frac{21}{64}$$

In the present paper, using contour integration technique, we will show that for negative integer r , sum (8) is solvable in closed form by

$$S(r) = -S\left(\frac{1}{2} - r\right) + \frac{4}{16^{-r}} \binom{-2r}{-r}^2 (H_{-r} - H_{-2r} + \log 2), \quad r = 0, -1, -2, \dots \quad (9)$$

where H_n are the harmonic numbers $H_n = \sum_{k=1}^n \frac{1}{k}$.

As a consequence of this result, in section 2, we derive the new representation for Saalschutzyan ${}_4F_3(1)$ series with special set of the parameters

$$\left(n - \frac{1}{2}\right) {}_4F_3\left(1, 1, n + \frac{1}{2}, n + \frac{1}{2}; 2, n + 1, n + 1; 1\right) =$$

$$\frac{4n^2}{2n-1} (H_{n-1} + \log 4) - \frac{16^n}{\binom{2n}{n}^2} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, n - \frac{1}{2}; 1, n + \frac{1}{2}; 1\right) \quad (10)$$

1. Evaluation

We consider two cases, when r is positive and negative. We denote $S^+(r) = S(r)$ for $\operatorname{Re}(r) > 0$ and $S^-(r) = S(r)$ for $\operatorname{Re}(r) \leq 0$.

Let r be a positive integer. We transform series (1) to a definite integral involving complete elliptic integrals. Multiplying the summand by x^{k+r} and differentiating it with respect to x , we get

$$g(r, x) = x^{r-1} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \frac{x^k}{16^k} = \frac{2}{\pi} x^{r-1} \mathbf{K}(x), \quad |x| < 1. \quad (11)$$

where $\mathbf{K}(x)$ is the elliptic integral. Integrating both sides of (11), we arrive at

$$S^+(r) = \int_0^1 g(r, x) dx = \frac{2}{\pi} \int_0^1 x^{r-1} \mathbf{K}(x) dx, \quad \operatorname{Re}(r) > 0. \quad (12)$$

In the next subsections we evaluate $S^+(r)$, by first developing a recurrent equation for $S^+(r)$, and then solving it by iteration. The result depends on the disparity of r .

Now let us consider the second case when r is a negative integer. We split the series $S(r)$ into two sums as

$$S^-(r) = \sum_{\substack{k=0 \\ k \neq -r}}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)16^k} = \left(\sum_{k=0}^{-r-1} + \sum_{k=-r+1}^{\infty} \right) \frac{\binom{2k}{k}^2}{(k+r)16^k}.$$

Leaving the first sum unchanged, and converting the second sum into an elliptic integral (by applying the same reasoning as above), we obtain

$$S^-(r) = \sum_{k=0}^{-r-1} \frac{\binom{2k}{k}^2}{(k+r)16^k} + \int_0^1 x^{r-1} \left(\frac{2}{\pi} \mathbf{K}(x) - \sum_{k=0}^{-r} \binom{2k}{k}^2 \frac{x^k}{16^k} \right) dx, \quad (13)$$

$$\operatorname{Re}(r) \leq 0$$

In subsection 1.3, using the contour integration technique, we establish a functional relation between $S^-(r)$ into $S^+(r)$.

1.1 $S^+(r)$ for r a non-negative integer

Consider the system of indefinite integrals

$$\begin{cases} k_p(x) = \int x^p \mathbf{K}(x) dx \\ e_p(x) = \int x^p \mathbf{E}(x) dx \end{cases} \quad (14)$$

where the parameter p is a positive integer or zero, and $\mathbf{E}(x)$ and $\mathbf{K}(x)$ are complete elliptic integrals. Using integration by parts, the above integral system can be reduced to the system of coupled recurrent equations

$$\begin{cases} k_p(x) = x^p k_0(x) - 2p(k_p(x) - k_{p-1}(x) + e_{p-1}(x)) \\ e_p(x) = x^p e_0(x) - \frac{2}{3}p(e_{p-1}(x) + e_p(x) + k_p(x) - k_{p-1}(x)) \end{cases}$$

with initial conditions

$$\begin{cases} 2k_0(x) = \mathbf{E}(x) + (x-1)\mathbf{K}(x) \\ \frac{3}{2}e_0(x) = (x+1)\mathbf{E}(x) + (x-1)\mathbf{K}(x) \end{cases}$$

Eliminating $e_{p-1}(x)$ from the first equation, and $k_{p-1}(x)$ and $k_p(x)$ from the second, the system is simplified to

$$\begin{cases} k_p(x) = \frac{4p^2}{(2p+1)^2} k_{p-1}(x) + \frac{2x^p \mathbf{E}(x) + 2(2p+1)(x-1)x^p \mathbf{K}(x)}{(2p+1)^2} \\ e_p(x) = \frac{4p^2}{(2p+1)(2p+3)} e_{p-1}(x) + \frac{2(1-2p+(2p+1)x)x^p \mathbf{E}(x) + 2(x-1)x^p \mathbf{K}(x)}{(2p+1)(2p+3)} \end{cases}$$

Now we compute the values of $k_p(x)$ and $e_p(x)$ at the limiting points $x=0$ and $x=1$. We get two recurrent equations

$$\begin{aligned} k_p(1) &= \frac{4p^2}{(2p+1)^2} k_{p-1}(1) + \frac{2}{(2p+1)^2}, \quad p \geq 1 \\ k_p(0) &= 0, \quad p \geq 0 \\ k_0(1) &= 2. \end{aligned} \tag{15}$$

and

$$\begin{aligned} e_p(1) &= \frac{4p^2}{(2p+1)(2p+3)} e_{p-1}(1) + \frac{4}{(2p+1)(2p+3)}, \quad p \geq 1 \\ e_p(0) &= 0, \quad p \geq 0 \end{aligned} \tag{16}$$

In view of formulas (12) and (15), we conclude that

$$S^+(r) = \frac{2}{\pi} (k_{r-1}(1) - k_{r-1}(0)) = \frac{2}{\pi} k_{r-1}(1) \tag{17}$$

where $S^+(r)$ satisfies the recurrence relation

$$\begin{aligned} \left(r + \frac{1}{2}\right)^2 S^+(r+1) - r^2 S^+(r) &= \frac{1}{\pi}, \quad r \geq 1 \\ S^+(1) &= \frac{4}{\pi}. \end{aligned} \tag{18}$$

Equation (18) can be solved by iteration (see section 3 for details). We have proven

Proposition 1.1 *Let n be a positive even. Then $S(n)$ defined by (1) evaluates to*

$$S(n) = \frac{16^n}{\pi n^2 \binom{2n}{n}} \sum_{k=0}^{n-1} \binom{2k}{k}^2 \frac{1}{16^k}. \quad (19)$$

1.2 $S^+(r)$ for r a positive half-integer

Consider slightly different (than (14)) system of indefinite integrals

$$\begin{cases} \hat{k}_p(x) = \int x^{p-\frac{1}{2}} \mathbf{K}(x) dx \\ \hat{e}_p(x) = \int x^{p-\frac{1}{2}} \mathbf{E}(x) dx \end{cases} \quad (20)$$

where the parameter p is a positive integer or zero, and $\mathbf{E}(x)$ and $\mathbf{K}(x)$ are complete elliptic integrals. Using integration by parts, we transform (20) to the system of recurrent equations

$$\begin{cases} p^2 \hat{k}_r(x) = (p - \frac{1}{2})^2 \hat{k}_{p-1}(x) + \frac{1}{2} x^{p-\frac{1}{2}} \left(\mathbf{E}(x) + 2p(x-1) \mathbf{K}(x) \right) \\ p(p+1) \hat{e}_r(x) = (p - \frac{1}{2})^2 \hat{e}_{p-1}(x) + x^{p-\frac{1}{2}} \left((p(x-1) + 1) \mathbf{E}(x) + \frac{x-1}{2} \mathbf{K}(x) \right) \end{cases} \quad (21)$$

where

$$\begin{aligned} \hat{k}_0(x) &= \pi \sqrt{x} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; x\right) \\ \hat{e}_0(x) &= \pi \sqrt{x} {}_3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; x\right) \end{aligned}$$

and ${}_3F_2(x)$ is the hypergeometric function. By computing the limits at $x = 0$ and $x = 1$, system (21) yields (assuming $p > 0$)

$$\begin{aligned} \hat{k}_p(1) &= \frac{(p - \frac{1}{2})^2}{p^2} \hat{k}_{p-1}(1) + \frac{1}{2p^2}, \quad p \geq 1 \\ \hat{k}_p(0) &= 0, \quad p \geq 0 \\ \hat{k}_0(1) &= 4G \end{aligned} \quad (22)$$

where G is Catalan's constant. Therefore,

$$S^+\left(p + \frac{1}{2}\right) = \frac{2}{\pi} \hat{k}_p(1), \quad p \geq 0 \quad (23)$$

The sequence $S^+(r)$, where r is a positive half integer, satisfies the same recurrence equation (18), but with a different initial condition:

$$\begin{aligned} \left(r + \frac{1}{2}\right)^2 S^+(r+1) - r^2 S^+(r) &= \frac{1}{\pi} \\ S^+\left(\frac{1}{2}\right) &= \frac{8G}{\pi} \end{aligned} \quad (24)$$

Solving this recurrence by iteration (see section 3 for details), we have proven

Proposition 1.2 *Let n be a positive integer. Then $S(n + \frac{1}{2})$ defined by (1) evaluates to*

$$S\left(n + \frac{1}{2}\right) = \frac{4}{\pi} \frac{\binom{2n}{n}^2}{16^n} \left(2G + \sum_{k=0}^{n-1} \frac{16^k}{\binom{2k}{k}^2 (2k+1)^2} \right) \quad (25)$$

1.3 $S^-(r)$, for r a negative integer

Recall formula (13). Observing that the finite sum inside of the integrand

$$\sum_{k=0}^{-r} \binom{2k}{k}^2 \frac{x^k}{16^k}$$

is the Taylor expansion of $\frac{2}{\pi} \mathbf{K}(x)$ at $x=0$, we pull that sum out of integration, by understanding integration in the Hadamard sense (*finite part*). Computing limits at the end points and obliterating logarithmic and polynomial order singularities, we get

$$S^-(r) = f.p. \frac{2}{\pi} \int_0^1 x^{-r-1} \mathbf{K}(x) dx.$$

Comparing this integral with formula (12) immediately implies that

$$S^-(r) = S^+(r) + F(r)$$

where $F(r)$ is an unknown function. The necessity of F becomes obvious once we recall that in the original series we skip the term $k = -r$, when r is a negative integer. In order to find F , we derive a contour integral representation for the sum $S(r)$ as

$$S(r) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s) \Gamma(\frac{1}{2}-s)}{\Gamma(1-s) \Gamma(\frac{1}{2}+s)} \frac{ds}{(r-s)}. \quad (26)$$

The contour $(\gamma - i\infty, \gamma + i\infty)$ is a straight line lying in the strip $0 < \gamma = \text{Re}(s) < \frac{1}{2}$. Integral (26) evaluates to (1) by summing residues at single poles $s = 0, -1, -2, \dots$, lying to the left of the contour. However, if r is a negative integer, the integrand in (26) has a double pole at $s = r$. According to the definition of $S(r)$ we must skip this pole. Thus, we have

$$S^-(r) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s) \Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)} \frac{ds}{(r-s)} - \text{res}_{s=r} \left(\frac{\Gamma(s) \Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)} \frac{1}{(r-s)} \right) \quad (27)$$

As a matter of fact, the contour integral herein can also be computed via residues at the poles $s = \frac{1}{2}, \frac{3}{2}, \dots$, lying to the right of the contour. Evaluating the integral via those poles allows us to avoid the double pole at $s = r$. This yields

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s) \Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)} \frac{ds}{(r-s)} = \\ & - \sum_{k=0}^{\infty} \frac{(2k)!^2}{k!^4 (k-r+\frac{1}{2}) 16^k} = -S^+\left(\frac{1}{2}-r\right). \end{aligned} \quad (28)$$

Finally, computing the residue

$$\text{res}_{s=r} \left(\frac{\Gamma(s) \Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)} \frac{1}{(r-s)} \right) = \frac{4}{16^{-r}} \binom{-2r}{-r}^2 \left(H_{-2r} - H_{-r} - \log 2 \right)$$

we establish

Proposition 1.3 *Let r be a negative integer or zero. Then*

$$S^-(r) = -S^+\left(\frac{1}{2}-r\right) - \frac{4}{16^{-r}} \binom{-2r}{-r}^2 \left(H_{-r} - H_{-2r} + \log 2 \right) \quad (29)$$

where $S^+(\frac{1}{2}-r)$ is defined in Proposition 1.2.

1.4 $S^-(r)$ for r a negative half integer

This case immediately follows from the previous subsection, taking into consideration that the integrand in (26) has only a single pole at $s = r$.

Proposition 1.4 *Let n be a positive integer. Then*

$$S^-\left(-n + \frac{1}{2}\right) = -S^+(n). \quad (30)$$

2. Special cases of hypergeometric functions

In this section we derive a particular case of the Saalschutzyan hypergeometric series ${}_4F_3(1)$. We begin by recalling that the hypergeometric series ${}_pF_p(a_1, \dots, a_{p+1}; b_1, \dots, b_p; 1)$ is called Saalschutzyan if parameters a_i and b_i satisfy the relation

$$1 + a_1 + a_2 + \dots + a_{p+1} = b_1 + \dots + b_p$$

Proposition 2.1 *Let n be a positive integer. Then*

$$\begin{aligned} \frac{(2n-1)^2}{8n^2} {}_4F_3\left(1, 1, n + \frac{1}{2}, n + \frac{1}{2}; 2, n+1, n+1; 1\right) = \\ -\frac{4G}{\pi} + H_{n-1} + \log 4 - \frac{2}{\pi} \sum_{k=0}^{n-2} \frac{16^k}{(2k+1)^2 \binom{2k}{k}^2} \end{aligned} \quad (31)$$

where G is Catalan's constant, and H_n are harmonic numbers.

Proof.

In view of formula (29) with $r = -n$, $n = 0, 1, 2, \dots$, we have

$$S^(-n) = -S^+\left(n + \frac{1}{2}\right) - \frac{4}{16^n} \binom{2n}{n}^2 \left(H_n - H_{2n} + \log 2\right) \quad (32)$$

where $S^+(n + \frac{1}{2})$ is defined in (25). On the other hand, if we evaluate the original sum (8) by means of the hypergeometric function, we obtain

$$\begin{aligned} S^(-n) = \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(k-n) 16^k} + \\ \frac{\binom{2n+2}{n+1}^2}{16^{n+1}} {}_4F_3\left(1, 1, n + \frac{3}{2}, n + \frac{3}{2}; 2, n+2, n+2; 1\right). \end{aligned} \quad (33)$$

The finite sum in the right-hand side of (33) can be evaluated in terms of harmonic numbers (see Proposition 3.2) as

$$16^n \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k (n-k)} = 4 \binom{2n}{n}^2 \sum_{k=0}^{n-1} \frac{1}{2k+1} = 2 \binom{2n}{n}^2 (2H_{2n-1} - H_{n-1}) \quad (34)$$

Combining formulas (32) and (33), and replacing n by $n-1$, we arrive at (31).

Remark. By using different ideas, formula (31) was first proved in [13].

3. Addendum

In this section we provide a solution to equations (18) and (24)

Proposition 3.1 *The solutions to the recurrence relation*

$$\begin{aligned} (2n+1)^2 x_{n+1} - (2n)^2 x_n &= a, \\ x_1 &= b \end{aligned} \quad (35)$$

is

$$x_n = \frac{16^n}{4n^2 \binom{2n}{n}^2} \left(b + a \sum_{k=1}^{n-1} \frac{\binom{2k}{k}^2}{16^k} \right) \quad (36)$$

Proof.

We solve recurrence (35) by iteration. Iterating it $n-1$ times, we get

$$x_{n+1} = b \prod_{j=0}^{n-1} \frac{(2n-2j)^2}{(2n+1-2j)^2} + a \sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1} (2n-2j)^2}{\prod_{j=0}^k (2n+1-2j)^2}. \quad (37)$$

In pretty straightforward manner the finite products herein can be converted to the binomial coefficients by using Euler's product representation for the Gamma function. We obtain

$$\prod_{j=0}^{n-1} \frac{(2n-2j)}{(2n-2j+1)} = \frac{4^{n+1}}{2(n+1) \binom{2n+2}{n+1}}$$

and

$$\sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1} (2n-2j)^2}{\prod_{j=0}^k (2n-2j+1)^2} = \frac{16^{n+1}}{4(n+1)^2 \binom{2n+2}{n+1}^2} \sum_{k=1}^n \frac{\binom{2k}{k}^2}{16^k}.$$

Substituting them into (37) yields the desired result.

Proposition 3.2 *Let n be a positive integer. Then*

$$\frac{16^n}{4 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k (n-k)} = \sum_{k=0}^{n-1} \frac{1}{2k+1}. \quad (38)$$

Proof.

We rearrange the terms in the sum in the left-hand side of (38), by summing them in the opposite order from $n-1$ to 0. We get

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(n-k) 16^k} = \sum_{k=1}^n \frac{\binom{2n-2k}{n-k}^2}{k 16^{n-k}}.$$

Since the summand evaluates to zero for $k > n$, we extend the range of summation to infinity. Using the definition of the hypergeometric series, we rewrite that sum in terms of ${}_4F_3$ as

$$\frac{16^n}{4 \binom{2n}{n}^2} \sum_{k=1}^{\infty} \frac{\binom{2n-2k}{n-k}^2}{k 16^{n-k}} = \frac{n^2}{(2n-1)^2} {}_4F_3\left(1, 1, 1-n, 1-n; 2, \frac{3}{2}-n, \frac{3}{2}-n; 1\right)$$

The latter further simplifies to polygamma functions by formula 7.5.3.43 from [12] as

$$\frac{2n^2}{(2n-1)^2} {}_4F_3\left(1, 1, 1-n, 1-n; 2, \frac{3}{2}-n, \frac{3}{2}-n; 1\right) = \psi\left(n + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) = \sum_{k=0}^{n-1} \frac{2}{2k+1}$$

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